DETAILED ALGORITHMS FOR OBJECT-IMAGE CORRESPONDENCE FOR CURVES UNDER CENTRAL AND PARALLEL PROJECTIONS

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Abstract. We provide details of the algorithms outlined in the paper posted on http://arxiv.org/abs/1004.0393

1. Formulas.


- Finite: \( P_f^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \) (1)

- Affine

\[
P_a^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

1.2. Differential Invariants. Let \( \gamma(t) = (x(t), y(t)) \) be a parametric curve, and let “dot” denote differentiation with respect to a parameter, then the lowest order invariants obtained via moving frame construction are:

- Euclidean: \( \kappa = \frac{\ddot{y}x - \dot{x}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \) and \( \kappa_s \), where \( ds = \sqrt{\dot{x}^2 + \dot{y}^2} \) dt.

- Equi-affine (SA(2)):

\[
\mu = \frac{3 \kappa (\kappa_{ss} + 3 \kappa^3) - 5 \kappa_s^2}{9 \kappa^{8/3}} \quad \text{and} \quad \mu_\alpha, \quad \text{where} \quad d\alpha = \kappa^{1/3} \, ds.
\]

- Projective (\( \mathcal{PG}\mathcal{L}(3) \)):

\[
\eta = \frac{6\mu_{\alpha\alpha\alpha} \mu_\alpha - 7 \mu_{\alpha\alpha}^2 - 9 \mu_\alpha^2 \mu}{6 \mu_\alpha^{4/3}} \quad \text{and} \quad \mu_\rho, \quad \text{where} \quad d\rho = \frac{\mu^{1/3}}{\mu_\alpha} \, d\alpha.
\]

We will use the following rational invariants:

- Affine rational invariants:

\[
J_a = \frac{(\mu_\alpha)^2}{\mu^3}, \quad K_a = \frac{\mu_{\alpha\alpha}}{3 \mu^2}.
\]

- Projective rational invariants:

\[
J_p = \eta^3, \quad K_p = \eta_\rho.
\]

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2. Algorithm for finite projections.

2.1. Outline of the algorithm.

Algorithm 1. (Outline for finite projections.)

INPUT: a planar curve $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, and a spatial curve $\Gamma(t) = (z_1(s), z_2(s), z_3(s))$, $s \in \mathbb{R}$, with rational parameterizations.

OUTPUT: YES or NO answer to the question ”Does there exist a finite projection [P], such that $[C_\gamma] = [P][C_\Gamma]$ is satisfied?”.

STEPS:
1. if $\exists c \in C$ then follow a special procedure, else
2. evaluate $PGL(3)$-invariants given by (5) on $\gamma(t)$. The result consists of two rational functions $J_p|_\gamma(t)$ and $K_p|_\gamma(t)$ of $t$;
3. for arbitrary $c_1, c_2, c_3 \in \mathbb{R}$ define a curve $c_{c_1,c_2,c_3}(s) = \left( \frac{z_1(s)+c_1}{z_3(s)+c_2}, \frac{z_2(s)+c_3}{z_3(s)+c_4} \right)$;
4. evaluate $PGL(3)$-invariants given by (5) on $c_{c_1,c_2,c_3}(s)$ – obtain two rational functions $J_p|_{c}(c_1, c_2, c_3, s)$ and $K_p|_{c}(c_1, c_2, c_3, s)$ of $c_1, c_2, c_3$ and $s$;
5. if $\exists c_1, c_2, c_3 \in \mathbb{R}$ s.t. the signatures $\mathcal{S}_\gamma = \{(J_p|_\gamma(t), K_p|_\gamma(t)) | t \in \mathbb{R}\}$ and $\mathcal{S}_s = \{(J_p|_{c}(c_1, c_2, c_3, s), K_p|_{c}(c_1, c_2, c_3, s)) | s \in \mathbb{R}\}$ coincide, then
   OUTPUT: YES, else OUTPUT: NO.

If the output is YES then, in many cases, we can, in addition to establishing the existence of $c_1, c_2, c_3$ in Step 8 of the algorithm, find at least one of such triplets explicitly. We then know that $C_\Gamma$ can be projected to $C_\gamma$ by a projection centered at $(-c_1, -c_2, -c_3)$.

We can also, in many cases, determine explicitly a transformation $[A] \in PGL(3)$ that maps $C_\gamma$ to $C_{c_1,c_2,c_3}$. We then know that $C_\Gamma$ can be projected to $C_\gamma$ by the projection $[P] = [A][P_f^0][B]$, where $P_f^0$ is defined by (1) and $B$ is defined by

$$B := \begin{pmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (6)

2.2. Detailed Algorithm.

Algorithm 2. (Detailed algorithm for finite projections.)

INPUT: a planar curve $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, and a spatial curve $\Gamma(t) = (z_1(s), z_2(s), z_3(s))$, $s \in \mathbb{R}$, with rational parameterizations.

OUTPUT: YES or NO answer to the question ”Does there exist a finite projection [P], such that $[C_\gamma] = [P][C_\Gamma]$ is satisfied?”.

STEPS:
1. If $\forall t: x''(t)y'(t) - y''(t)x'(t) = 0$ then
   - compute the Euclidean curvature $K(s)$ for the curve $\Gamma$;
   - if $\forall s: K(s) = 0$ then OUTPUT: YES and exit the procedure, else
     - compute the Euclidean torsion $T(s)$ for the curve $\Gamma$;
     - if $\forall s: T(s) = 0$ then OUTPUT: YES, else OUTPUT: NO
   - exit the procedure;
else proceed to the next step.
2. For arbitrary real 
\( c_1, c_2, c_3 \) define a curve 
\( \epsilon_{c_1, c_2, c_3}(s) = \left( \frac{z_1(s) + c_1}{z_3(s) + c_3}, \frac{z_2(s) + c_2}{z_3(s) + c_3} \right) \).

3. Use (3) to evaluate the cube of \( S A \)-curvature 
\( \mu^3|_{\gamma}(t) \). The result is a rational function of \( t \).

4. If \( \exists \ m \in \mathbb{R} \) s. t. \( \forall t \in \mathbb{R} : \mu^3|_{\gamma}(t) = m \) then
   \( \bullet \) use (3) to evaluate \( \mu^3|_{\epsilon_{c_1, c_2, c_3}}(s) \) (the result is a rational function of \( c_1, c_2, c_3 \) and \( s \));
   \( \bullet \) if \( \exists \ a, c_1, c_2, c_3 \in \mathbb{R} \) s. t. \( \forall s \in \mathbb{R} \)
   \[ \mu^3|_{\epsilon_{c_1, c_2, c_3}}(s) = a \]  \tag{7}
   then \( \text{OUTPUT} : \text{YES} \), else \( \text{OUTPUT} : \text{NO} \);
   \( \bullet \) exit the procedure.\nelse proceed to the next step.

5. Evaluate \( PGL(3) \)-invariants (5) on \( \gamma(t) \). Obtain two rational functions of \( t \), \( J_p|_{\gamma}(t) \) and \( K_p|_{\gamma}(t) \).

6. Evaluate \( PGL(3) \)-invariants (5) on \( \epsilon_{c_1, c_2, c_3}(s) \). Obtain two rational functions \( J_p|_{\epsilon}(c_1, c_2, c_3, s) \)
   and \( K_p|_{\epsilon}(c_1, c_2, c_3, s) \) of \( c_1, c_2, c_3 \) and \( s \).

7. If \( \exists j, k \in \mathbb{R} \) s. t. \( \forall t \in \mathbb{R} : J_p|_{\gamma}(t) = j \) and \( K_p|_{\gamma}(t) = k \), then
   \( \bullet \) if \( \exists c_1, c_2, c_3 \in \mathbb{R} \) s. t. \( \forall s \in \mathbb{R} \),
   \[ J_p|_{\epsilon}(c_1, c_2, c_3, s) = j \quad \text{and} \quad K_p|_{\epsilon}(c_1, c_2, c_3, s) = k, \]  \tag{8}
   then \( \text{OUTPUT} : \text{YES} \), else \( \text{OUTPUT} : \text{NO} \);
   \( \bullet \) exit the procedure;\nelse proceed to the next step.

8. If \( \exists c_1, c_2, c_3 \in \mathbb{R} \), such that \( \forall t \in \mathbb{R} \), where denominators of \( J_p|_{\gamma}(t) \) and \( K_p|_{\gamma}(t) \) are non-zero,
   \( \exists s \in \mathbb{R} \):
   \[ J_p|_{\epsilon}(c_1, c_2, c_3, s) = J_p|_{\gamma}(t) \quad \text{and} \quad K_p|_{\epsilon}(c_1, c_2, c_3, s) = K_p|_{\gamma}(t), \]  \tag{9}
   then \( \text{OUTPUT} : \text{YES} \), else \( \text{OUTPUT} : \text{NO} \).

**What happens at each step:** In the first step, we consider a possibility that a given planar curve \( \gamma(t) \) is a line. A spatial curve \( \Gamma(s) \) can be projected to a straight line if and only if it is a planar curve. To determine whether \( \Gamma(s) \) is planar curve or not, we can use its Euclidean curvature and torsion. If the curvature is zero, then the image \( C_{\Gamma} = \{ \Gamma(s), s \in \mathbb{R} \} \) is a straight line and the torsion is undefined. If torsion is defined and identically zero, then \( C_{\Gamma}(s) \) is a planar curve, but not a line. In both cases \( \Gamma(s) \) can be projected to a straight line\(^1\) and the output is "yes". Conversely, a curve in \( \mathbb{R}^3 \) that is projected to a straight line lies in the plane defined by its image and the center of the projection. So if \( \Gamma \) is not a planar curve it can not be projected onto a straight line and the output is "no".

If \( \gamma(t) \) is not a line, we proceed to step 2, where we define a family of planar curves \( \epsilon_{c_1, c_2, c_3}(s) \). The goal is to decide if there are some values of the parameters \( c_1, c_2, c_3 \) for which \( \epsilon_{c_1, c_2, c_3}(s) \) is \( PGL(3) \)-equivalent to \( \gamma(t) \). From our projection criteria it follows that this provide necessary and sufficient conditions for \( \Gamma \) can be projected to \( \gamma \) by a central projection.

In step 3, we compute \( \mu^3|_{\gamma} \) – the cube of the \( S A \)-curvature of \( \gamma(t) \).

In step 6, we consider a possibility that \( \gamma(t) \) is a part of a conic, or equivalently its \( S A \)-curvature \( \mu \) is a constant. All conics on the plane are \( PGL(3) \)-equivalent and therefore if there exist \( a, c_1, c_2, c_3 \in \mathbb{R} \) such \( \mu^3|_{\epsilon_{c_1, c_2, c_3}}(s) = a \) the output is "yes", otherwise it is "no".

\(^{1}\)If \( C_{\Gamma} \) is a straight line, then call it \( L \) and choose the center of projection not on \( L \). Let \( H \) be the plane containing \( C_{\Gamma} \) and the center of the projection. If \( C_{\Gamma} \) is a planar curve, but not a straight line, then let \( H \) be the plane containing \( C_{\Gamma} \). There exists a line \( L \) that intersects \( C_{\Gamma} \) at two points. Choose the center of the projection on \( L \) but not on \( C_{\Gamma} \).

In both cases, let the image plane be parallel to \( L \) and not parallel to \( H \). Then the image of \( C_{\Gamma} \) is a straight line, which equals to the intersection of \( H \) with the image plane. By an appropriate choice of coordinates on the image plane, we can make the image to have the same defining equation as \( C_{\Gamma} \).
If we proceed to step 5 then \( \gamma \) is not \( PGL(3) \)-exceptional and we can compute its \( PGL(3) \)-invariants \( J_p|\gamma(t) \) and \( K_p|\gamma(t) \).

In step 6, we compute \( PGL(3) \)-invariants \( J_p|\gamma|c_1, c_2, c_3, s \) and \( K_p|\gamma|c_1, c_2, c_3, s \) for the curve \( \epsilon_{c_1,c_2,c_3}(s) \).

In step 7, we address a possibility that \( J_p|\gamma(t) \) and \( K_p|\gamma(t) \) are both constant. Then if \( \exists c_1, c_2, c_3 \in \mathbb{R} \), such that \( J_p|\gamma|c_1, c_2, c_3, s \), \( K_p|\gamma|c_1, c_2, c_3, s \) are the same constants then the algebraic curves \( C_\gamma \) and \( C_{c_1,c_2,c_3}(s) \) are \( PGL(3) \)-equivalent. From our projection criteria it follows that \( \Gamma \) can be projected to \( \gamma \) by a central projection. If there is no \( c_1, c_2, c_3 \in \mathbb{R} \) satisfying this condition then \( \Gamma \) can not be projected to \( \gamma \) by a central projection.

We arrive to step 8, if the signature of \( \gamma \), given by \( \mathcal{S}_\gamma = \{(J_p|\gamma(t), K_p|\gamma(t))|t \in \mathbb{R}\} \), is a one-dimensional subset of \( \mathbb{R}^2 \). If the conditional statement of this step is satisfied then there exist \( c_1, c_2, c_3 \) such that \( \mathcal{S}_\gamma \subset \mathcal{S}_\epsilon \), where \( \mathcal{S}_\epsilon = \{(J_p|\epsilon|c_1, c_2, c_3, s), K_p|\epsilon|c_1, c_2, c_3, s))|s \in \mathbb{R}\} \). It follows that algebraic curves \( C_\gamma \) and \( C_{c_1,c_2,c_3}(s) \) are \( PGL(3) \)-equivalent. From our projection criteria it follows that \( \Gamma \) can be projected to \( \gamma \) by a central projection. If the conditional statement is not satisfied then \( \Gamma \) can not be projected to \( \gamma \) by a central projection.

**Computational challenges:** Steps 7 and 8 are the only computational challenging steps of the algorithm. They involve real quantifier elimination – a problem which is known to be algorithmically solvable [6]. There is an extensive body of literature devoted to computationally effective methods in real quantifier elimination, including [1], [2], [4], [3], [5].

3. Algorithm for affine projections.

3.1. Outline of the algorithm.

**Algorithm 3.** (Outline for affine projections.)

**INPUT:** a planar curve \( \gamma(t) = (x(t), y(t)), t \in \mathbb{R}, \) and a spatial curve \( \Gamma(t) = (z_1(s), z_2(s), z_3(s)) \), \( s \in \mathbb{R}, \) with rational parameterizations.

**OUTPUT:** YES or NO answer to the question "Does there exist an affine projection \([P], \) such that \([C_\gamma] = [P]|[C_\Gamma] \) is satisfied?".

**STEPS:**

1. if \( \gamma \) is \( \mathcal{A}(2) \)-exceptional (a straight line or a parabola) then follow a special procedure, else
2. evaluate \( \mathcal{A}(2) \)-invariants given by (4) on \( \gamma(t) \). The result consists of two rational functions \( J_{a|\gamma}(t) \) and \( K_{a|\gamma}(t) \) of \( t \);
3. define a curve \( \alpha(s) = (z_2(s), z_3(s)) \);
4. evaluate \( \mathcal{A}(2) \)-invariants given by (4) on \( \alpha(s) \) – obtain two rational functions \( J_{a|\alpha}(s) \) and \( K_{a|\alpha}(s) \) of \( s \);
5. if \( \mathcal{S}_\gamma \setminus \{J_{a|\gamma}(t), K_{a|\gamma}(t)\} \mid t \in \mathbb{R} \} \) and \( \mathcal{S}_\alpha \setminus \{J_{a|\alpha}(s), K_{a|\alpha}(s)\} \mid s \in \mathbb{R} \} \) coincide, then
   **OUTPUT:** YES and exit the procedure, else
6. for arbitrary \( b \in \mathbb{R} \) define a curve \( \beta_0(s) = (z_1(s) + b z_2(s), z_3(s)) \);
7. evaluate \( \mathcal{A}(2) \)-invariants given by (4) on \( \beta_0(s) \) – obtain two rational functions \( J_{a|\beta}(b,s) \) and \( K_{a|\beta}(b,s) \) of \( b \) and \( s \);
8. if \( \exists b \in \mathbb{R} \) s. t. the signatures \( \mathcal{S}_\gamma \setminus \{J_{a|\gamma}(t), K_{a|\gamma}(t)\} \mid t \in \mathbb{R} \} \) and \( \mathcal{S}_\beta \setminus \{J_{a|\beta}(b,s), K_{a|\beta}(b,s)\} \mid s \in \mathbb{R} \} \) coincide, then
   **OUTPUT:** YES and exit the procedure, else
9. for arbitrary \( c, f \in \mathbb{R} \) define a curve \( \delta_{c,f}(s) = (z_1(s) + c z_3(s), z_2 + f z_3(s)) \);
10. evaluate \( \mathcal{A}(2) \)-invariants given by (4) on \( \delta_{c,f}(s) \) – obtain two rational functions \( J_{a|\delta}(c,f,s) \) and \( K_{a|\delta}(c,f,s) \) of \( c, f \) and \( s \);
11. if \( \exists c, f \in \mathbb{R} \) s. t. the signatures \( \mathcal{S}_\gamma \setminus \{J_{a|\gamma}(t), K_{a|\gamma}(t)\} \mid t \in \mathbb{R} \} \) and \( \mathcal{S}_\beta \setminus \{J_{a|\beta}(c,f,s), K_{a|\beta}(c,f,s)\} \mid s \in \mathbb{R} \} \) coincide, then
Although the algorithm for affine projections includes more steps than its finite projection counterpart, it is computationally less challenging. If the output is YES then, in many cases, we can find an affine projection explicitly.

References


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