Inductive Approach to Cartan's Moving Frame Method

Irina Kogan
Yale University

July, 2002

Generalized definition of a moving frame

M. Fels and P.J. Olver (1999)

A moving frame is an equivariant smooth map $\rho\colon M \to G$.

$$\begin{tikzcd}
G \arrow[r, "R_{g^{-1}}"] & G \\
M \arrow[u, "\rho"] \arrow[r, "g"'] & M \arrow[u, "\rho"']
\end{tikzcd}$$

Theorem.
A moving frame exists if the action is regular and free.

A (local) moving frame $\Longleftrightarrow$ free action and (local) cross-section $\mathcal{K}$:

- $\mathtt{codim}\,\mathcal{K} = \dim \mathcal{O}_z$,
- $\mathcal{K}$ is transversal to $\mathcal{O}_z$ for $\forall z \in U \subset M$,
- $\mathcal{K} \cap \mathcal{O}_z$ consists of at most one point.

Define $\rho(z)$, by the condition $\rho(z) \cdot z \in \mathcal{K}$

$$\Downarrow$$

$$\begin{aligned} \rho(g \cdot z) g \cdot z &= \rho(z) \cdot z \\ &\Downarrow \text{ freeness} \\ \rho(g \cdot z) &= \rho(z) g^{-1} \end{aligned}$$

Invariantization of functions $f \colon M \to \mathbb{R}$

$$\iota(f)(z) = f(\rho(z) \cdot z)$$

$Z^i = \iota(z^i) \supset$ a complete set of functionally independent invariants.

Euclidean Geometry on the plane.

The Frénet Frame.

$SE(2) = SO(2) \ltimes \mathbb{R}^2$

Infinitesimal arc-length $ds = \sqrt{1 + u_x^2}\, dx$

$$T = \left(\frac{dx}{ds}, \frac{du}{ds} \right), \qquad \frac{dT}{ds} = \kappa N$$

Basic differential invariants: the Euclidean curvature κ and its derivatives $\kappa_s = \frac{d\kappa}{ds}$, κ_{ss}, etc.

Affine Geometry on the plane.

$SA(2) = SL(2) \ltimes \mathbb{R}^2$

Infinitesimal arc-length $d\alpha = u_{xx}^{1/3}\, dx$

$$T = \left(\frac{dx}{d\alpha}, \frac{du}{d\alpha} \right), \qquad N = \frac{dT}{d\alpha} \qquad \Longrightarrow |T \times N| = 1$$

$$\frac{dN}{d\alpha} = \mu T$$

Basic differential invariants: affine curvature μ and its derivatives $\mu_\alpha = \frac{d\mu}{d\alpha}$, $\mu_{\alpha\alpha}$, etc.

Moving Frames on Homogeneous Spaces.

P.A. Griffiths (1974), M.L.Green (1978)

$$\begin{tikzcd} & G \arrow[d] \\ N \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & G/H \end{tikzcd}$$

Euclidean Example:

$$\tilde{f} = \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & -\frac{u_x}{\sqrt{1+u_x^2}} & x \\ \frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & u \\ 0 & 0 & 1 \end{pmatrix} = (T, N, X).$$

$$\left(\frac{dT}{ds}, \frac{dN}{ds}, \frac{dX}{ds}\right) = (T, N, X) \begin{pmatrix} 0 & -\kappa & 1 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{f}^{-1} \frac{d}{ds}(\tilde{f})\, ds = \begin{pmatrix} 0 & -\kappa ds & ds \\ \kappa ds & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Jet spaces:

$J^k = J^k(M,p)$ are bundles over M.
The fiber over $z \in M$ consists of the equivalence classes of p-dim. submanifolds with k-th order contact at z.

Local coordinates on J^k:

$$\{x^1, \dots, x^p, u^1, \dots u^q, u^\alpha_J\}, \text{ where}$$
$$J = (j_1, \dots, j_p) \qquad |J| = j_1 + \dots + j_p \leq k, \quad j_i \geq 0.$$

Projections:

$$J^\infty \dots \to J^k \to J^{k-1} \to \dots J^1 \to J^0 = M$$

$$\pi^n_k \colon J^t \to J^k \text{ for } t \geq k$$

Jets of submanifolds:

If $N : u^\alpha = f^\alpha(\mathbf{x})$ then

$$j^k(N) : u^\alpha = f^\alpha(\mathbf{x}), \qquad u^\alpha_J = \frac{\partial^k f^\alpha}{\partial^{j_1} x^1 \dots \partial^{j_p} x^p}.$$

Prolongation of the action:

$$g \cdot j^k(N) = j^k(g \cdot N).$$

Theorem. (Ovsyannikov, Olver)
The action of G is loc. effective on open subsets

$$\Downarrow$$

$\exists n \leq \dim G$ (the order of stabilization), s. t. the prolonged action is loc. free on an open dense $\mathcal{V}^n \subset J^n$.

$$\Downarrow$$

$\exists$ moving frame $\rho : \mathcal{V}^n \mapsto G$

ρ can be lifted to J^k for $k > n$:

$$\rho(z^{(k)}) = \rho\left(\pi^k_n(z^{(k)})\right)$$

$$\Downarrow$$

- complete set of differential invariants,
- invariant differential forms,
- invariant differential operators.

Euclidean action on the plane:

$$\begin{aligned} x &\mapsto y = \cos(\phi)x - \sin(\phi)u + a \\ u &\mapsto v = \sin(\phi)x + \cos(\phi)u + b \\ u_x &\mapsto v_1 = \frac{\sin(\phi) + \cos(\phi)u_x}{\cos(\phi) - \sin(\phi)u_x} \\ u_{xx} &\mapsto v_2 = \frac{u_{xx}}{(\cos(\phi) - \sin(\phi)u_x)^3} \\ u_{xxx} &\mapsto v_3 = \frac{(\cos(\phi) - \sin(\phi)u_x)u_{xxx} + 3\sin(\phi)u_{xx}^2}{(\cos(\phi) - \sin(\phi)u_x)^5} \end{aligned}$$

cross-section on J^1:

$$\mathcal{K}_E = \{x = 0,\, u = 0,\, u_x = 0\}$$

$$\Downarrow$$

moving frame is the solution of

$$y = 0,\, v = 0,\, v_1 = 0 :$$

$$\phi = -\arctan(u_x), \qquad a = -\frac{x + u_x u}{\sqrt{u_x^2 + 1}}, \qquad b = \frac{u_x x - u}{\sqrt{u_x^2 + 1}}$$

Substitute in v_2 and v_3:

$$\begin{aligned} I_2^e &= \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \\ I_3^e &= \kappa_s = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3} \end{aligned}$$

$$\begin{aligned} dy &= \mathsf{cos}(\phi)dx - \mathsf{sin}(\phi)du = \\ &\quad (\mathsf{cos}(\phi) - \mathsf{sin}(\phi)u_x)\,dx - \mathsf{sin}(\phi)\theta = d_H y + d_V y, \end{aligned}$$

where $\theta = du - u_x\,dx$

$$d_H y = (\mathsf{cos}(\phi) - \mathsf{sin}(\phi)u_x)\,dx, \qquad d_V y = -\mathsf{sin}(\phi)\theta$$

moving frame $\Rightarrow \phi = -\mathsf{arctan}(u_x)$

$$d_H y \Rightarrow ds = \sqrt{1 + u_x^2}\,dx$$

Modifications:

- **Recursive algorithm (some conditions on the group action)**
  - Construct differential invariants on J^k
    order by order,
    at each step normalizing more and more of the group parameters
    at the end obtaining a moving frame for the group G.

    $$\Downarrow$$

    The structure of invariant differential forms on J^k.

- **Inductive algorithm**
  $G = AB$ s. t. $A \cap B$ is discrete.
  - invariants and moving frames for A and B can be used to construct invariants and a moving frame for G.

    $$\Downarrow$$

    Relations among the invariants of G and its subgroups.

Inductive algorithm:

$$G = BA, \qquad B \cap A \text{ is discrete}$$

Proposition 1.
$\exists\, n_a$ and a local c.-s. $\mathcal{K}_A \subset J^{n_a}$ s. t.

- A acts loc. freely near $\mathcal{K}_A$,

- $\mathcal{K}_A$ is invariant under the action of B.

$$\Downarrow$$

- m.f. $\rho_A \colon J^{n_a} \longrightarrow A$ defined by
$$\rho_A(z^{(n_a)}) \cdot z^{(n_a)} \in \mathcal{K}_A.$$

- for $k \geq n_a$ the action of B is well defined on:
$$\mathcal{K}_A^k = \{z^{(k)} \in J^k |\, \pi_{n_a}^k(z^{(k)}) \in \mathcal{K}_a\}.$$

Proposition 2.

- $\exists\, n$ s. t. B acts loc. freely on

$$\mathcal{K}_A^n = \{z^{(k)} \in J^n |\, \pi^n_{n_a}(z^{(k)}) \in \mathcal{K}_A\}.$$

- $\exists$ c.-s. $\mathcal{K} \subset \mathcal{K}_a^n$ for B-action.

$$\Downarrow$$

m.f. $\rho_B \colon \mathcal{K}_A^n \to B$ defined by $\rho_B(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}$

Proposition 3. $\mathcal{K}$ is a c.-s. for the G-action.

$$\Downarrow$$

$\rho_G \colon J^n \to G$ is a m.f. for G.

$$\boxed{\rho_G(z^{(n)}) = \rho_B(\rho_A(z^{(n)}) \cdot z^{(n)})\rho_A(z^{(n)})}$$

$$\boxed{\rho_B(\rho_A(z^{(\infty)}) \cdot z^{(\infty)})\rho_A(z^{(\infty)}) \cdot z^{(\infty)}}$$

contains a complete set of G- invariants.

From the Euclidean to the affine action:

$$SA(2,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 = B \cdot SE(2,\mathbb{R})$$

$$B = \left\{ \begin{pmatrix} \tau & \lambda \\ 0 & \frac{1}{\tau} \end{pmatrix} \right\}$$

where B is the isotropy group of

$$z_0^{(1)} = \{x = 0, u = 0, u_x = 0\} = \mathcal{K}_E.$$

$\dim SA(2,\mathbb{R}) = 5 \Rightarrow$ lowest inv. on J^4.

Step 1: Prolong the action of B to J^4:

$$\begin{aligned}
x &\rightarrow \tau x + \lambda u, \\
u &\rightarrow \frac{1}{\tau} u, \\
u_x &\rightarrow \frac{u_x}{\tau(\tau + \lambda u_x)}, \\
u_{xx} &\rightarrow \frac{u_{xx}}{(\tau + \lambda u_x)^3}, \\
u_{xxx} &\rightarrow \frac{(\tau + \lambda u_x) u_{xxx} - 3\lambda u_{xx}^2}{(\tau + \lambda u_x)^5}, \\
u_{xxxx} &\rightarrow \frac{(\tau + \lambda u_x)^2 u_{xxxx} - 10(\tau + \lambda u_x)\lambda u_{xx} u_{xxx} + 15\lambda^2 u_{xx}^3}{(\tau + \lambda u_x)^7}.
\end{aligned}$$

Step 2 Restrict these transformations to

$$\begin{aligned}\mathcal{K}_E^4 &= \{z^{(4)} | \pi_1^4(z^{(4)}) = z_0^{(1)}\} \\ &= \{z^{(4)} | x = 0, u = 0, u_x = 0\}:\end{aligned}$$

$$\begin{aligned}u_{xx} &\to \frac{u_{xx}}{\tau^3}, \\ u_{xxx} &\to \frac{\tau u_{xxx} - 3\lambda u_{xx}^2}{\tau^5}, \\ u_{xxxx} &\to \frac{\tau^2 u_{xxxx} - 10\tau\lambda u_{xx}u_{xxx} + 15\lambda^2 u_{xx}^3}{\tau^7}.\end{aligned}$$

Step 3 Choose a cross-section for B action

$$\mathcal{K}^{(4)} = \{z^{(4)} \in \mathcal{K}_E^4 | u_{xx} = 1, u_{xxx} = 0\}$$

$$\Downarrow$$

M. f. $\rho_B \colon \mathcal{K}_E^4 \to B$:

$$\tau = (u_{xx})^{1/3} \text{ and } \lambda = \frac{u_{xxx}}{3(u_{xx})^{5/3}}.$$

$$\Downarrow$$

invariant for B action on $\mathcal{K}_E^4$:

$$I_4^b = \frac{u_{xx}u_{xxxx} - \frac{5}{3}(u_{xxx})^2}{(u_{xx})^{8/3}}.$$

Step 4 Substitute Euclidean invariants:

$$\mu = I_4^a = \frac{I_2^e I_4^e - \frac{5}{3}(I_3^e)^2}{(I_2^e)^{8/3}}$$

In terms of κ :

$$I_2^e = \kappa,\; I_3^e = \kappa_s,\; I_4^e = \kappa_{ss} + 3\kappa^3$$

$$\Downarrow$$

$$\boxed{\mu = \frac{\kappa(\kappa_{ss}+3\kappa^3)-\frac{5}{3}\kappa_s^2}{\kappa^{8/3}}.}$$

Differentiation with respect to the affine arc length:

$$d\alpha = \kappa^{1/3} ds$$

$$\Downarrow$$

$$\boxed{\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}}\frac{d}{ds}}$$

Affine invariant differential operator:

$$x \to y = \tau x + \lambda u \Rightarrow d_H y = (\tau + \lambda u_x)\, dx$$

B-moving frame:

$$\tau = (u_{xx})^{1/3} \quad \text{and} \quad \lambda = \frac{u_{xxx}}{3(u_{xx})^{5/3}}.$$

$$\Downarrow$$

$$\left(u_{xx}^{1/3} \quad + \quad \frac{u_{xxx} u_x}{3(u_{xx})^{5/3}} \right) dx.$$

Invariantization with respect to the Euclidean action.

Note that on $\mathcal{K}_E^4$:

$$u_x = 0,\ u_{xx} = I_2^e = \kappa,\ u_{xxx} = I_3^e = \kappa_s,\ dx = ds$$

$$\Downarrow$$

$$d\alpha = \kappa^{1/3} ds$$

$$\Downarrow$$

$$\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}} \frac{d}{ds}$$

From the affine to the projective action:

$$PSL(3,\mathbb{R}) \text{ acts on } \mathbb{R}^2 :$$

$$x \mapsto \frac{\alpha x + \beta u + \gamma}{\delta x + \epsilon u + \zeta}; \qquad u \mapsto \frac{\lambda x + \nu u + \tau}{\delta x + \epsilon u + \zeta}$$

$$PSL(3,\mathbb{R}) = B \cdot SA(2,\mathbb{R})$$

$$B = \left\{ \begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & \frac{1}{a} \end{pmatrix} \right\}$$

is the isotropy group of

$$\mathcal{K}_A = z_0^{(3)} = \{x = 0, u = 0, u_1 = 0, u_2 = 1, u_3 = 0\}$$

$$\mathcal{K}_A^7 = \{z^{(7)} | \pi_3^7(z^{(7)}) = z_0^{(3)}\}$$

- restrict the prolonged B-action to $\mathcal{K}_A^7$.

- choose cross-section for B-action to $\mathcal{K}_A^7$:

$$\mathcal{K} = \{z^{(7)} \in \mathcal{K}_A^7 | u_4 = 0, u_5 = 1, u_6 = 0\}$$

- normalize parameters a, b and c and substitute affine invariants:

Projective curvature in terms of affine invariants:

$$\eta = \frac{-7\mu_{\alpha\alpha}^2 + 6\mu_\alpha\mu_{\alpha\alpha\alpha} - 3\mu\mu_\alpha^2}{6\mu_\alpha^{8/3}}$$

Invariant differential operator for the projective action in terms of the affine:

$$\frac{d}{d\varrho} = \frac{1}{\mu_\alpha^{1/3}}\frac{d}{d\alpha}$$

The moving frame for the projective group is the product of the matrices:

$$\begin{pmatrix} 1 & -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha} & 0 \\ 0 & \mu_\alpha^{1/3} & 0 \\ -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_\alpha^{4/3}} & \frac{1}{18}\frac{\mu_{\alpha\alpha}^2-3\mu\mu_\alpha^2}{\mu_\alpha^{7/3}} & \frac{1}{\mu_\alpha^{1/3}} \end{pmatrix} \times$$

$$\begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Affine and Euclidean action for curves in $\mathbb{R}^3$

$SA(3,\mathbb{R}) = B \cdot SE(3,\mathbb{R})$ and $B \cap SA(3,\mathbb{R}) = I$, where

$$B = \begin{pmatrix} a & b & c \\ 0 & f & g \\ 0 & 0 & \frac{1}{af} \end{pmatrix}.$$

is the isotropy group of

$$\mathcal{K}_E = \{z^{(2)} |\, x = 0, u = 0, v = 0, u_x = 0, v_x = 0, v_{xx} = 0\}$$

$$\begin{aligned} I_5^{u,a} \quad &= \tfrac{1}{36}\,\kappa^{-4}\tau^{-7/2} \\ &\big(36\kappa^2\tau^2(\tau\kappa_{sss} - \kappa_{ss}\tau_s + 4\kappa^2\tau\kappa_s - \kappa\tau^2\tau_s - 3\kappa^3\tau_s + 2\tau^3\kappa_s) + \\ &60\tau^2\kappa(\kappa_s^2\tau_s - 3\tau\kappa_{ss}\kappa) - 6\kappa^2\tau(\tau_s^2\kappa_s - 3\kappa\tau_{ss}\tau_s) + \\ &160\kappa_s^3\tau^3 - 25\kappa^3\tau_s^3\big)\,, \end{aligned}$$

$$\begin{aligned} I_5^{v,a} = \quad &\tfrac{1}{24}\,\kappa^{-8/3}\,\tau^{-7/3}\,\big(36\kappa^2\tau^2(\kappa^2+\tau^2) - 20\kappa_s^2\tau^2 - 8\kappa\tau\kappa_s\tau_s \\ &- 35\kappa^2\tau_s^2 + 12\kappa\tau(\tau\kappa_{ss} + 2\kappa\tau_{ss})\big)\,. \end{aligned}$$

$$\boxed{d\alpha = (\kappa^2\,\tau)^{1/6}\,ds}$$

Recursive algorithm:

G acts on M regularly but not freely

$$\Updownarrow$$

$\exists$ a cross-section $\mathcal{K}_0$.

$$\rho_0 : M \to G : \rho_0(z) \cdot z \in \mathcal{K}_0.$$

is defined up to the isotropy groups on $\mathcal{K}$.

Coordinates of $\rho_0(z) \cdot z \supset$ a complete set of 0-th order invariants.

Example:

$G = SO(3, \mathbb{R})$ acting on $\mathbb{R}^3$.

$\mathcal{K}_0$ is the upper half z-axis

0-th order invariant: $r = \sqrt{x^2 + y^2 + z^2}$

Each point on $\mathcal{K}_0$ has the same isotropy group $H \simeq SO(2, \mathbb{R})$

$$[\rho_0] : M \longrightarrow H \backslash G$$

Definition: *Isotypic slice* is a c.-s. with the same isotropy group at each point.

$\exists$ isotypic slice
$\Rightarrow$ all orbits are of the same type
$\Leftarrow$ + proper action (e.g. G is compact).

Algorithm

(0) $\exists \mathcal{K}_0 \subset M$ with isotropy group $H_0 \subset G$

$$\Downarrow$$

$[\rho_0] : M \to H_0 \backslash G$ is G-equivariant.

H_0 acts on $\mathcal{K}_0^1 = \{z^{(1)} | \, \pi_0^1(z^{(1)}) \in \mathcal{K}_0\}$.

(1) $\exists \mathcal{K}_1 \subset \mathcal{K}_0^1$ with isotropy group $H_1 \subset H_0$

$$\Downarrow$$

$[\tau_1] : \mathcal{K}_0^1 \to H_1 \backslash H_0$ is H_0-equivariant.

$[\rho_1](z^{(1)}) = \left[\tau_1(\rho_0(z^{(1)}) \cdot z^{(1)})\rho_0(z^{(1)})\right] :$
$J^1 \to H_1 \backslash G$ is G-equivariant.

Coordinates of $\rho_1(z^{(1)})z^{(1)} \supset$ a complete set of 1-st order invariants.

$$\begin{tikzcd}
J^n \arrow[r, "{([\rho_n], \iota_n)}"] \arrow[d, "\pi^n_{n-1}"'] & (H_n \backslash G) \times \mathcal{K}_n \arrow[d, "{(\delta^n_{n-1}, \pi^n_{n-1})}"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
J^2 \arrow[r, "{([\rho_2], \iota_2)}"] \arrow[d, "\pi^2_1"'] & (H_2 \backslash G) \times \mathcal{K}_2 \arrow[d, "{(\delta^2_1, \pi^2_1)}"] \\
J^1 \arrow[r, "{([\rho_1], \iota_1)}"] \arrow[d, "\pi^1_0"'] & (H_1 \backslash G) \times \mathcal{K}_1 \arrow[d, "{(\delta^1_0, \pi^1_0)}"] \\
M \arrow[r, "{([\rho_0], \iota_0)}"'] & (H_0 \backslash G) \times \mathcal{K}_0
\end{tikzcd}$$

$H_{k-1} \supset H_k$ and $\pi^k_{k-1}(\mathcal{K}_k) = \mathcal{K}_{k-1}$

- G-equivariant maps:
  - π^k_{k-1} – jet projections $J^{k+1} \longrightarrow J^k$,
  - $\rho_k \colon J^k \longrightarrow H_k\backslash G$,
  - $\delta^k_{k-1}([g]_k) = [g]_{k-1} \colon H_k\backslash G \longrightarrow H_{k-1}\backslash G$. where $[g]_k$ is the image of g under $G \longrightarrow H_k\backslash G$.
- G-invariant map:

$$\iota_k(z^{(k)}) = \rho_k(z^{(k)}) \cdot z^{(k)} \colon J^k \longrightarrow \mathcal{K}_k.$$