Differential and Variational Calculus in Invariant Frames

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Outline:

- Group invariant contact structure on a jet bundle
- Structure equations for invariant frame
- Invariant variational complex
- Invariant Euler-Lagrange operator
- Invariant Noether correspondence
- Symbolic implementation

The talk is based on:

- *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, I. Kogan, P. Olver, Acta Appl. Math. 76, 137-193, (2003)
- Ongoing work with Ian Anderson

Group invariant contact structure on a jet bundle

Standard local coframe on $J^\infty(M,p)$

Local coordinates: $x^1,\dots,x^p,u^1,\dots,u^q,u^m_J$, $m=1\dots q$, J – multi-index.

Basis of horizontal sub-bundle	Basis of vertical sub-bundle

Cotangent

horizontal one-forms $d\,x^1,\dots,d\,x^p$	contact one-forms $\theta^m = du^m - \sum_{i=1}^p u^m_i\,dx^i,$ $\theta^m_J = du^m_J - \sum_{i=1}^p u^m_{Ji}\,dx^i.$

Tangent

total derivatives: $\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{m=1}^q u^m_i \frac{\partial}{\partial u^m}$ $+\sum_{m,J} u^m_{Ji}\frac{\partial}{\partial u^m_J}$	vertical derivatives $\frac{\partial}{\partial u^m},$ $\frac{\partial}{\partial u^m_J},\ m=1\dots q$

Prolongation of the action

$G \curvearrowright M \Rightarrow G \curvearrowright J^k(M, p)$ unique prolongation s.t

- G-action respects projections $J^k \longrightarrow J^l$, $\forall k > l$.
- g^*(contact form) $=$ contact form $\forall g \in G$.

Thm. (*Ovsiannikov*) $\exists n < \dim G$ s. t. G acts locally free on an open dense subset of $J^n(M, p)$.

Local invariants

$G \curvearrowright J^\infty \Rightarrow h\colon \mathfrak{g} \longrightarrow \mathcal{X}(J^\infty)$
$Im(h) = \Gamma$ infinitesimal generators of G-action.

- invariant functions $f(\mathbf{x}, \mathbf{u}^{(n)})$ (differential invariants):
  $\mathcal{L}_v f := v(f) = 0,\ \forall v \in \Gamma.$
- invariant vector fields: $Y \in \mathcal{X}(J^\infty)$: $\mathcal{L}_v X = 0,\ \forall v \in \Gamma.$
- invariant differential forms: $\lambda \in \Lambda^*(J^\infty)$: $\mathcal{L}_v \lambda = 0,\ \forall v \in \Gamma.$

$\exists G$- invariant contact structure on $\mathcal{U} \subset J^\infty(M,p)$.

Invariant horizontal basis	Invariant vertical basis

Cotangent

invariant "horizontal" one-forms $\omega^1, \dots, \omega^p$ $\tilde{H} = span\left\{\omega^i\right\} \neq H = span\left\{d\,x^i\right\}$ unless the action is projectable	invariant contact one-forms ϑ^m_J $\mathcal{C} = span\left\{\vartheta^m_J\right\} = span\left\{\theta^m_J\right\}$

Tangent

invariant total diff. operators $\mathcal{D}_1, \dots \mathcal{D}_p$ $span\left\{\mathcal{D}_i\right\} = span\left\{\frac{d}{dx_i}\right\}$	invariant vertical diff. operators $V_1, \dots, V_q, V^J_m, m = 1..q$ $span\left\{V_m\right\} \neq span\left\{\frac{\partial}{\partial u^m}\right\}$ unless the action is projectable

Example $SE(2)$-invariant contact structure on $J^\infty(\mathbb{R}^2, 1)$.

Cotangent

invariant "horizontal" one-form	invariant contact one-forms
$\omega = ds + \frac{u_x}{\sqrt{1+u_x^2}}\theta$, where $ds = \sqrt{1+u_x^2}dx$	$\vartheta = \frac{\theta}{\sqrt{1+u_x^2}}$ $\vartheta_x = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$ $\ldots$

Tangent

invariant total diff. operator:	invariant vertical diff. operators
$\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}}\frac{d}{dx} = \frac{d}{ds}$	$V = -\frac{u_x}{\sqrt{1+u_x^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+u_x^2}}\frac{\partial}{\partial u}$ $V^x = (1+u_x^2)\frac{\partial}{\partial u_x} + 3\,u_x u_{xx}\frac{\partial}{\partial u_{xx}}$ $\ldots$

Motivation

(*S. Lie:*)

Most of the symmetric problems can be written in terms of invariants.
It is natural to perform further computations in terms of invariants.

Example: Lagrangian invariant under $SE(2) = SO(2) \ltimes \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2\,(1+u_x^2)^{5/2}} dx = \int L dx \qquad \Longleftrightarrow \qquad \int \tfrac{1}{2}\kappa^2\, ds$$

$$\Big\downarrow\; E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right) \frac{\partial}{\partial u_x} \qquad\qquad ?? \qquad \Big\downarrow$$

$$E(L) = 0 \qquad\qquad \Longleftrightarrow \quad \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

$$E(L) =$$

$$\frac{1}{2}\,\frac{30\,{u_2}^3\,{u_1}^2 - 5\,{u_2}^3 - 20\,u_2\,u_1\,u_3 - 20\,u_2\,{u_1}^3\,u_3 + 2\,u_4 + 4\,u_4\,{u_1}^2 + 2\,u_4\,{u_1}^4}{(1+{u_1}^2)^{(9/2)}}.$$

$(\ u_1 = u_x, \ldots, u_4 = u_{xxxx})$

Invariantization: moving frame construction

(*Fels, Olver 1999*)

Local cross-section: submanifold $\mathcal{K}$ of open $\mathcal{U} \subset M$, s. t.

- $T_{\mathbf{z}}\mathcal{K}| \bigoplus T|_{\mathbf{z}}\mathcal{O}_{\mathbf{z}} = T|_{\mathbf{z}}\mathcal{U}, \quad \forall \mathbf{z} \in \mathcal{K}$
- $\mathcal{K}$ intersects each connected component of $\mathcal{O}_{\mathbf{z}} \cap \mathcal{U}$ at the unique point.

Thm (follows from the Frobenius thm)

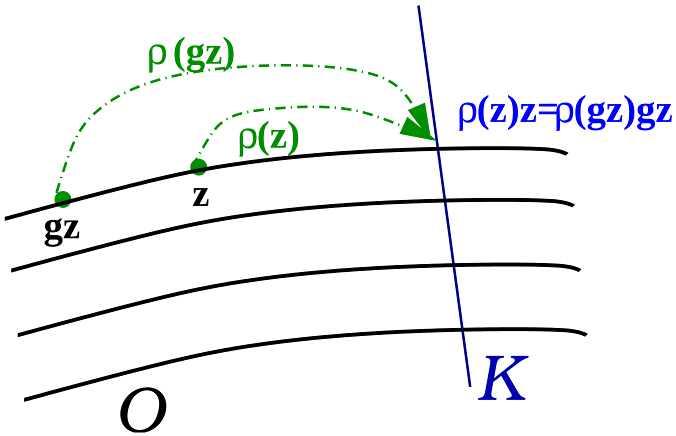
$G \curvearrowright \mathcal{U}$ semi. reg ($\Leftrightarrow \exists r, \forall \mathbf{z} \in \mathcal{U} : \dim \mathcal{O}_{\mathbf{z}} = r$.)

$$\Downarrow$$

$\exists$ loc. cross-section on $\mathcal{U}$

Moving frame map

$\rho : \mathcal{U} \longrightarrow G$ is defined by the condition $\rho(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}$



$G \curvearrowright M$ free $\Rightarrow \forall \mathbf{z} \in M : \dim \mathcal{O}_{\mathbf{z}} = \dim G = \kappa.$

$$\Downarrow$$

$\rho : \mathcal{U} \longrightarrow G$ defined by $\boxed{\rho(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}}$ smooth, G-equivariant:

$$\rho(g \cdot \mathbf{z}) \cdot (g \cdot \mathbf{z}) = \rho(\mathbf{z}) \cdot \mathbf{z}, \overset{freeness}{\Longrightarrow} \rho(g \cdot \mathbf{z}) = \rho(\mathbf{z}) g^{-1}.$$

Invariantization in terms of $\rho : \mathcal{U} \longrightarrow G$.

$$\mathbf{z}_0 = \mathcal{O}_{\mathbf{z}} \cap \mathcal{K} = \rho(\mathbf{z}) \cdot \mathbf{z}$$

Thm: The following maps define a projection on the corresponding invariant subspaces.

- functions: $\iota\colon \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}^G(\mathcal{U})$:

$$\iota(f)(\mathbf{z}) := f(\rho(\mathbf{z}) \cdot \mathbf{z}) = f(\mathbf{z}_0)$$

- differential forms: $\iota\colon \Lambda(\mathcal{U}) \longrightarrow \Lambda^G(\mathcal{U})$, $\quad \forall \lambda \in \Lambda^*(\mathcal{U}), \forall \mathbf{z} \in \mathcal{U}$:

$$\iota(\lambda)|_{\mathbf{z}} := \rho(\mathbf{z})^*(\lambda|_{\mathbf{z}_0}).$$

- vector fields: $\iota\colon \mathcal{X}(\mathcal{U}) \longrightarrow \mathcal{X}^G(\mathcal{U})$, $\quad \forall X \in \mathcal{X}(\mathcal{U}), \forall \mathbf{z} \in \mathcal{U}$:

$$\iota(X)|_{\mathbf{z}} = \left[\rho(\mathbf{z})^{-1}\right]_* X|_{\mathbf{z}_0}.$$

Properties of ι:

- ι preserves linear independence of forms and vector fields.
- ι -preserves pairing: $\langle \iota\lambda; \iota X \rangle =< \lambda; X >$.
- ι-preserves duality of bases.
- $\forall \mathbf{z} \in \mathcal{K}$, $\forall \lambda \in \Lambda^*(\mathcal{U})$, $\forall X \in \mathcal{X}(\mathcal{U})$:

$$\iota(f)|_{\mathbf{z}_0} = f|_{\mathbf{z}_0}, \iota(X)|_{\mathbf{z}_0} = X|_{\mathbf{z}_0}, \iota(\lambda)|_{\mathbf{z}_0} = \lambda|_{\mathbf{z}_0}.$$

- ι preserves contact ideal.

Structure equations for invariantized forms.

$$\boxed{d\,\iota\,(\lambda) = \iota\,(d\,\lambda) + \sum_{j=1}^{r} \rho^*(\mu^j) \wedge \iota\left[\mathcal{L}_{v_j}\,(\lambda)\right]}$$

$\dim G = \dim \mathcal{O} = r$

$\hat{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_r)^T$ a basis of $\mathfrak{g}$;

$\mu = (\mu^1, \ldots, \mu^r)$– dual basis of invariant differential forms on G (Maurer-Cartan forms)

$\mathbf{v} = (v_1, \ldots, v_r)^T$ – basis for infinitesimal generators of G-action

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Let zero set $K = (K^1(z), \ldots, K^r(z)) : \mathcal{U} \longrightarrow \mathbb{R}^r$ define $\mathcal{K}$.

Transversality condition $\det \mathbf{v}(K) := \mathcal{L}_{v_i}(K^j) \neq 0 \bmod K$.

$$\boxed{\rho^*(\mu) = -\iota\left[dK \cdot \mathbf{v}(K)^{-1}\right]}$$

- $\mathcal{K}$ – constructive ($\exists$ coordinate cross-section)
- ρ – non-constructive

How much can be done without ρ? $\Rightarrow$

Invariantization: implicit approach

Theorem. $\forall f \in \mathcal{F}(\mathcal{U}), \quad \forall \lambda \in \Lambda^*(\mathcal{U}), \quad \forall X \in \mathcal{X}(\mathcal{U})$
$\exists!$ locally invariant $\iota f \in \mathcal{F}^G(\mathcal{U}), \quad \iota\lambda \in (\Lambda^*)^G(\mathcal{U}), \quad \iota X \in \mathcal{X}^G(\mathcal{U})$ s.t.

$$\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}, \quad \iota X|_{\mathcal{K}} = X|_{\mathcal{K}}, \quad \iota\lambda|_{\mathcal{K}} = \lambda|_{\mathcal{K}}.$$

Structure equations

$$d\,\iota\,(\lambda) = \iota\,[d\,\lambda - (dK)A \wedge \mathbf{v}(\lambda)]$$
$$\text{where } A = \mathbf{v}(K)^{-1},$$
$$\mathbf{v}(\lambda) = (\mathcal{L}_{v_1}\lambda, \ldots, \mathcal{L}_{v_r}\lambda)^T$$

$\exists!$locally G- invariant frame/coframe on $\mathcal{U} \subset J^\infty(M,p)$ which coincide with the standard frame/coframe on $\mathcal{K}$.

Invariant horizontal basis	Invariant vertical basis

Cotangent

invariant "horizontal" one-forms $\omega^1, \ldots, \omega^p$	invariant contact one-forms ϑ^m_J $\mathcal{C} = span\,\{\vartheta^m_J\} = span\,\{\theta^m_J\}$

Tangent

invariant total diff. operators $\mathcal{D}_1 = \iota \frac{d}{dx_1}, \ldots, \mathcal{D}_p = \iota \frac{d}{dx_p}$ $span\,\{\mathcal{D}_i\} = span\left\{\frac{d}{dx_i}\right\}$	invariant vertical diff. operators $V_1, \ldots, V_q, V^J_m, m = 1..q$

Input:

- group action: $\mathbf{v} = (\mathbf{v}_1, \ldots \mathbf{v}_l)^T$ – basis for inf. generators.
- local cross-section: $\mathcal{K}$ – zero set $K(\mathbf{z}) = (K^1, \ldots, K^r) \colon \mathcal{J} \longrightarrow \mathbb{R}^r$,

$$\det \mathbf{v}(K) \neq 0 \bmod K.$$

Output (via linear algebra and differentiation)

- structure equations for invariant coframe and frame $\Rightarrow$ *structure of the differential algebra of invariants*
- prolongations of vector fields
- invariant integration by parts $\Rightarrow$ Euler-Lagrange and Helmholtz operators

Variational Bicomplex:

Gelfand, Tulczyjew, Tsujishita, Vinogradov, Takens, Anderson

Bigrading of exterior differential algebra:

Grading: $\Lambda^* = \bigoplus \Lambda^k$, where $\Lambda^k = \left\{ \sum \underbrace{\text{1-form} \wedge \cdots \wedge \text{1-form}}_{k \text{ times}} \right\}$.

$d\colon \Lambda^k \longrightarrow \Lambda^{k+1}, \quad d \circ d = 0 \Rightarrow$ de Rham complex:

Bigrading: $\Lambda^* = \bigoplus \Lambda^{s,t}$, where $\Lambda^{s,t} =$

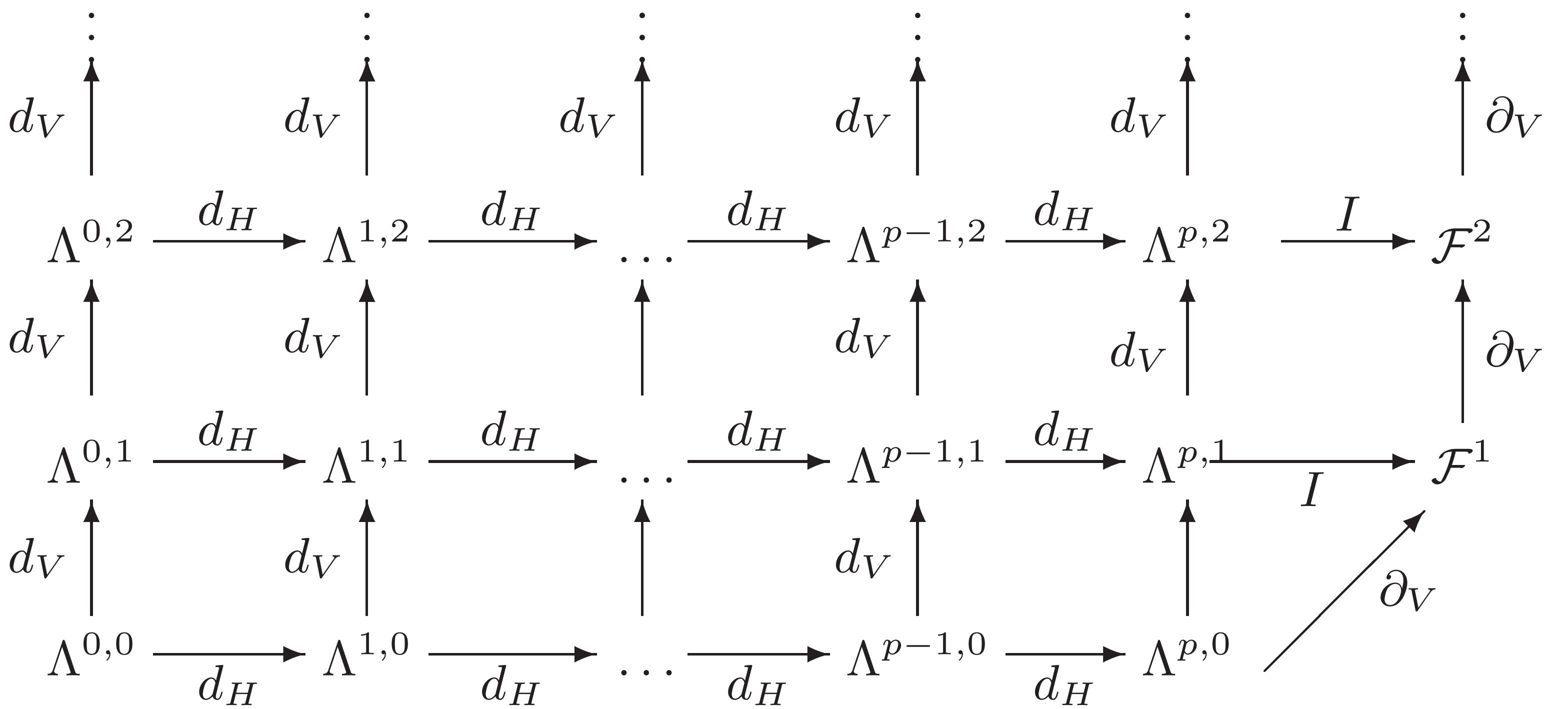
$$\left\{ \sum \underbrace{\text{hor. 1-form} \wedge \cdots \wedge \text{hor. 1-form}}_{s \text{ times}} \wedge \underbrace{\text{cont. 1-form} \wedge \cdots \wedge \text{cont. 1-form}}_{t \text{ times}} \right\}$$

$d\colon \Lambda^{s,t} \longrightarrow \Lambda^{s+1,t} \oplus \Lambda^{s,t+1} \Rightarrow d = d_H + d_V \Rightarrow$ Bicomplex:

$d^2 = (d_H + d_V)^2 = 0 \Rightarrow d_H^2 = 0,\ d_V^2 = 0,\ d_H \circ d_V = -d_V \circ d_H$

Example: $du = u_x dx + (du - u_x dx) = d_H u + d_V u$

Variational Bicomplex (locally exact)



- $d_V d_H = -d_H d_V, \quad d_H^2 = 0, \quad d_V^2 = 0$
- $\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) dx^1 \wedge \cdots \wedge dx^p \in \Lambda^{p,0}$-Lagrangian
- $I \colon \Lambda^{p,s} \longrightarrow \mathcal{F}^s \subset \Lambda^{p,s}$ - int. by parts operator, $I \circ d_H = 0$, $I^2 = I$
- $\partial_V = I \circ d_V$- variational derivative, $\partial_V \circ d_H = 0$

Integration by parts operator, $\lambda \in \Lambda^{p,s}$:

$$I(\lambda) = \sum_m \frac{1}{s}\theta^m \wedge \left(\left(\frac{\partial}{\partial u^m} \lrcorner \lambda \right) + \sum_J \left(-\frac{d}{dx_J} \right) \left(\frac{\partial}{\partial u^m_J} \lrcorner \lambda \right) \right).$$

Euler-Lagrange operator ($\lambda \in \Lambda^{p,0}$)

$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)})d\mathbf{x} \xrightarrow{d_V} \sum_{m,J} \frac{\partial L}{\partial u^m_J}\theta^m_J \wedge d\mathbf{x} \xrightarrow{I} \sum_m EL^m(\mathbf{x}, \mathbf{u}^{(\mathbf{2n})})\theta^m \wedge d\mathbf{x}$

$\partial_V = I \circ d_V \Rightarrow E^m(\mathbf{x}, \mathbf{u}^{(\mathbf{2n})}) = 0, \quad m = 1, \ldots, q$ – Euler-Lagrange eq.

$$d_V(\lambda) - \partial_V(\lambda) = d_H(\mu),\ \mu \in \Lambda^{p-1,1}$$

$\Lambda^{p,0} \supset$ trivial Lagrangians $= \ker \partial_V \supset d_H \Lambda^{p-1,0}$

Local exactness: $\ker \partial_V = d_H \Lambda^{p-1,0}$

Inverse Problem of Calculus of Variations

Is a given system of equations $\{EL^m(\mathbf{x}, \mathbf{u^{(2n)}}) = 0 | m = 1, \ldots, q\}$ equivalent to an Euler-Lagrange system ?

Simplification: for $\beta = \sum_m EL^m(\mathbf{x}, \mathbf{u^{(2n)}})\theta^m \wedge d\mathbf{x} \in \Lambda^{p,1}$
$\exists ? \lambda \in \Lambda^{p,0}$, s. t. $\partial_V(\lambda) = \beta$

Euler-Lagrange forms $= Im(\partial_V^0 : \Lambda^{p,0} \longrightarrow \Lambda^{p,1}) \subset Ker(\partial_V^1 : \Lambda^{p,1} \longrightarrow \Lambda^{p,2})$

Local exactness: $\Rightarrow \ker \partial_V^1 = Im(\partial_V^0)$

Noether Correspondence.

Symmetries of $\int L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \leftrightarrow$ Conservation laws of $E(L)$

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generalized divergence symmetry: $Y = \sum_{j=1}^{q} Q^j(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial u^j}$ s.t

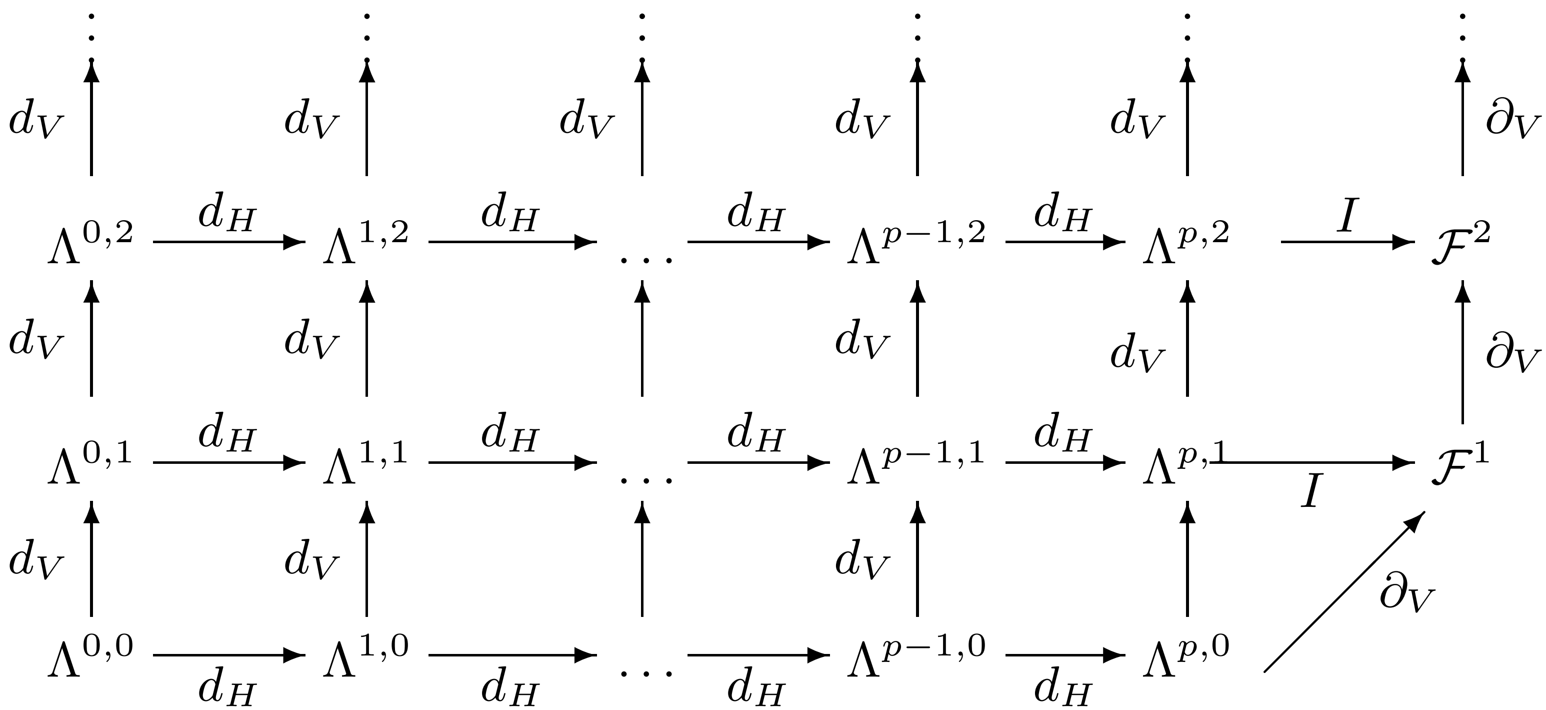
$\exists A = (A_1, \ldots, A_p)$: $pr^n Y(L) = \texttt{Div} A$

$$\Downarrow$$

conservation law: $P = (P_1, \ldots, P_p)$ s.t. $\texttt{Div} P \equiv 0 \text{ mod } E(L)$

E.L.-eq: $E(L) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L\left(\mathbf{x}, (\mathbf{u} + \varepsilon f)^{(n)}\right) - \texttt{Div} M$

Q, A, $M \Rightarrow P$



- $\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) dx^1 \wedge \cdots \wedge dx^p \in \Lambda^{p,0}$-Lagrangian
- $d_V \lambda - \partial_V \lambda = d_H \mu, \;\; \mu \in \Lambda^{p-1,1}$
- Y is a variational symmetry: $\mathcal{L}_{prY}(\lambda) = d_H(\alpha)$, $\alpha \in \Lambda^{p-1,0}$
- $\pi = Y \lrcorner \mu + \alpha \in \Lambda^{p-1,0}$ conservation law: $d_H \pi = 0$ mod E-L eq.

Example:

$$\boxed{\lambda = \tfrac{1}{2}\kappa^2 ds \;\Rightarrow\; EL = \kappa_{ss} + \tfrac{1}{2}\kappa^3}$$

Generalized symmetry: $Y = \frac{-3\,u_2^2\,u_1 + u_3\,(1+u_1^2)}{(1+u_1^2)^{\frac{5}{2}}}\frac{\partial}{\partial u} = \kappa_s\,\sqrt{1+u_1^2}\,\frac{\partial}{\partial u}$.

Conservation law:

$$\begin{aligned} P &= \frac{1}{4}\,\frac{u_2^4 + 36\,u_2^4\,u_1^2 - 24\,u_2^2\,u_3\,u_1 - 24\,u_2^2\,u_3\,u_1^3 + 4\,u_3^2 + 8\,u_3^2\,u_1^2 + 4\,u_3^2\,u_1^4}{(1+u_1^2)^6} \\ &= \frac{1}{2}\kappa_s^2 + \frac{1}{8}\kappa^4 \end{aligned}$$

Check:

$$\texttt{Div}P = \frac{dP}{dx} = \kappa_s\,\sqrt{1+u_1^2}EL$$

conservation law with characteristic

$$Q = \kappa_s\,\sqrt{1+u_1^2}$$

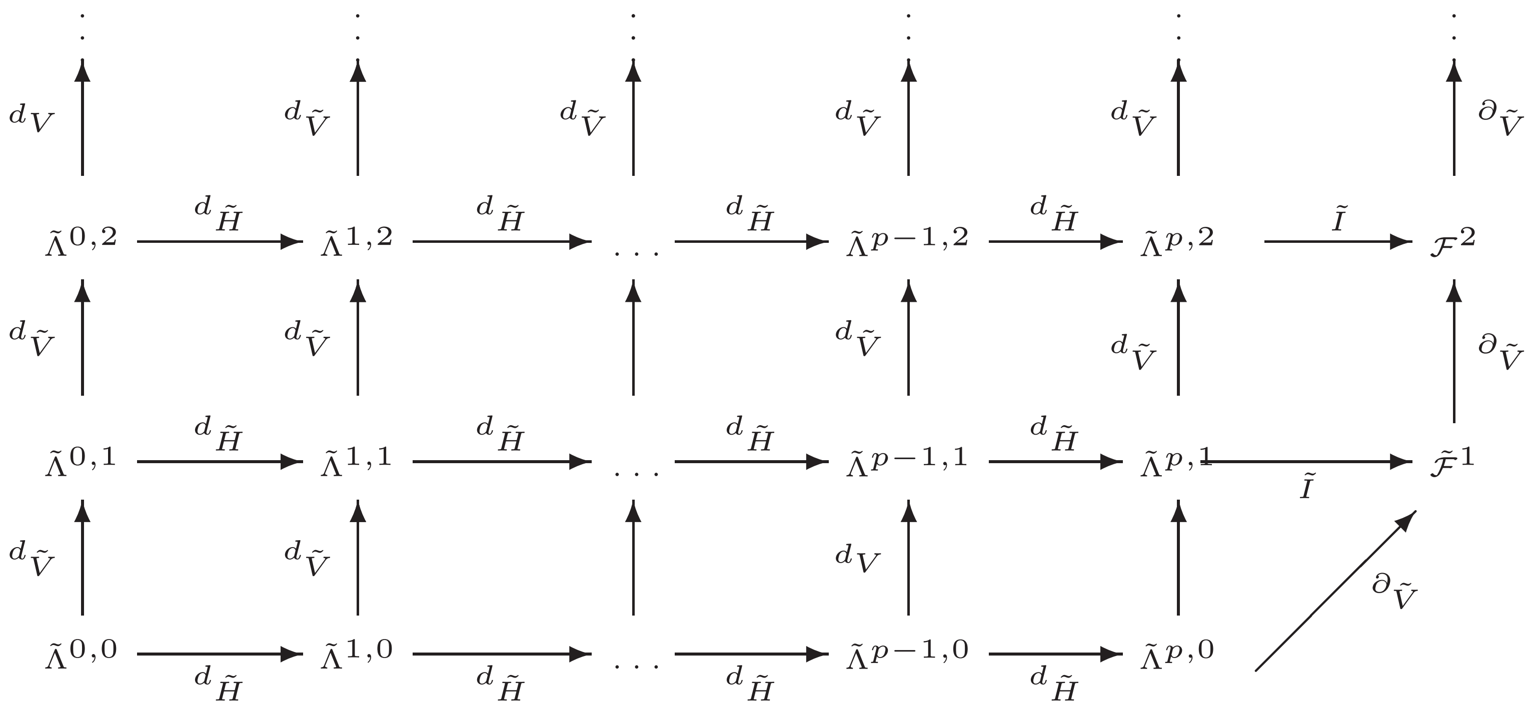
Invariant Variational Complex

Invariant local coframe/frame on $\mathcal{U} \subset J^\infty(E,p)$.

Invariant horizontal basis	Invariant vertical basis
Invariants moving coframe	
invariant "horizontal" one-forms $\omega^1, \ldots, \omega^p$ $\tilde{H} = span\left\{\omega^i\right\} \neq H = span\left\{d\,x^i\right\}$ unless the action is projectable	invariant contact one-forms $\vartheta^m, \vartheta^m_J$ $span\left\{\vartheta^m, \vartheta^m_J\right\} = span\left\{\theta^m, \theta^m_J\right\}$
Invariants moving coframe	
invariant total diff. operators $\mathcal{D}_1, \ldots \mathcal{D}_p$ $span\left\{\mathcal{D}_i\right\} = span\left\{\frac{d}{dx_i}\right\}$	invariant vertical diff. operators $V_1, \ldots, V_q, V^J_m, m = 1..q$ $span\left\{V_m\right\} \neq span\left\{\frac{\partial}{\partial u^m}\right\}$ unless the action is projectable

Invariant Variational “Bicomplex”.

• variational edge is a complex • interior rows are locally exact.



- $\lambda \in \tilde\Lambda^{p,0}$-Lagrangian $\longrightarrow \partial_{\tilde V}\lambda = d_{\tilde V}\lambda - d_{\tilde H}\mu$, $\mu \in \tilde\Lambda^{p-1,1}$ (E.-L. eq.)
- variational symmetry: $Y \lrcorner d_{\tilde V}\lambda = d_{\tilde H}(\alpha)$, $\alpha \in \tilde\Lambda^{p-1,0}$
- $\pi = Y \lrcorner \mu + \alpha \in \tilde\Lambda^{p-1,0}$ conservation law: $d_{\tilde H}\pi = 0$ mod E.-L. eq.

Invariant bigrading: $\Lambda^* = \bigoplus \tilde{\Lambda}^{s,t} = \bigoplus \iota\Lambda^{s,t}$.

$\tilde{\Lambda}^{s,t} \neq \Lambda^{s,t}$ (**)

$$\boxed{d\,\iota\,(\lambda) = \iota\,[d\,\lambda - (dK)A \wedge \mathbf{v}(\lambda)]\,,\ \text{where } A = \mathbf{v}(K)^{-1}.}$$

$\lambda \in \Lambda^{s,t} \Rightarrow \iota\lambda \in \tilde{\Lambda}^{s,t}$ and

$\mathcal{L}_v(\lambda) = [\mathcal{L}_v(\lambda)]_0 + [\mathcal{L}_v(\lambda)]_V \in \Lambda^{s,t} \bigoplus \Lambda^{s-1,t+1}$ (**)

$$d = d_{\tilde{H}} + d_{\tilde{V}} + d_W\,(**)$$

where

$$\begin{aligned} d_{\tilde{H}}\,\iota\,(\lambda) &= \iota\,[d_H\,\lambda - (d_H K)A \wedge \mathbf{v}(\lambda)_0]\,, \\ d_{\tilde{V}}\,\iota\,(\lambda) &= \iota\,[d_V\,\lambda - (d_V K)A \wedge \mathbf{v}(\lambda)_0]\,, \\ d_W\,\iota\,(\lambda) &= \iota\,[-(d_V K)A \wedge \mathbf{v}(\lambda)_V]\,, \end{aligned}$$

(**) unless the action is projectable.

for $s \geq 1$ unless the action is projectable:

$$d\colon \tilde{\Lambda}^{s,t} \longrightarrow \tilde{\Lambda}^{s+1,t} \oplus \tilde{\Lambda}^{s,t+1} \oplus \tilde{\Lambda}^{s-1,t+2} \Rightarrow d = d_{\tilde{H}} + d_{\tilde{V}} + d_W$$

$$d^2 = (d_{\tilde{H}} + d_{\tilde{V}} + d_W)^2 = 0$$

$$\boxed{d_{\tilde{H}}^2 = 0, \quad d_{\tilde{V}}^2 + d_{\tilde{H}} d_W + d_W d_{\tilde{H}} = 0, \quad d_{\tilde{H}} \circ d_{\tilde{V}} = -d_{\tilde{V}} \circ d_{\tilde{H}}, \quad d_W^2 = 0}$$

$$\boxed{\partial_{\tilde{V}}^2 = 0}$$

Euler-Lagrange operator in the invariant bicomplex

Example: Lagrangian invariant under $SE(2) = SO(2) \ltimes \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2\,(1+u_x^2)^{5/2}}dx = \int L dx \quad \Longleftrightarrow \quad \int \tfrac{1}{2}\kappa^2\, ds$$

$$\Big\downarrow \; E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_x} \qquad ?? \qquad \Big\downarrow$$

$$E(L) = 0 \quad \Longleftrightarrow \quad \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

$$E(L) = \frac{1}{2}\,\frac{30\,{u_2}^3\,{u_1}^2 - 5\,{u_2}^3 - 20\,u_2\,u_1\,u_3 - 20\,u_2\,{u_1}^3\,u_3 + 2\,u_4 + 4\,u_4\,{u_1}^2 + 2\,u_4\,{u_1}^4}{(1+{u_1}^2)^{(9/2)}}.$$

$(\ u_1 = u_x, \ldots, u_4 = u_{xxxx})$

G-invariant variational problems for curves in $\mathbb{R}^2$.

$$\int \tilde{\lambda} = \int \tilde{L}\,(\kappa, \kappa_1, \ldots, \kappa_n)\,\omega,$$

$\kappa-$ fundamental invariant under $G-$ action

$\omega-$ invariant arc-length

$\kappa_i = \mathcal{D}^i\kappa-$ i-the derivative of κ with respect to invariant arc-length

$\mathcal{D}$ derivative with respect to ω

The invariant Euler-Lagrange operator:

$$\boxed{\mathcal{A}^*\mathcal{E} - B^*\mathcal{H}}$$

$$\mathcal{E}\left(\tilde{L}\right) = \sum_{i=0}^{n} (-\mathcal{D})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}\left(\tilde{L}\right) = \sum_{i>j\geq 0}^{n} \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$d_{\tilde{V}}(\omega) = \mathcal{B}(\vartheta_0) \wedge \omega, \quad d_{\tilde{V}}(\kappa) = \mathcal{A}(\vartheta_0)-$ depend on G, computable by diff. and linear algebra.

for multi-dimensional formula see Kogan, Olver 2003

Invariant integration by parts for plane curves.

$G \curvearrowright J(\mathbb{R}^2, 1)$, ω – invariant differential form, $\mathcal{D}$ – dual invariant differential operator, κ – generating invariant, $\kappa_i = (\mathcal{D})^i \kappa$.

$$\tilde{\lambda} = \tilde{L}\,(\kappa, \kappa_1, \ldots, \kappa_n)\,\omega$$

$$d_{\tilde{V}}\tilde{\lambda} = d_{\tilde{V}}\tilde{L} \wedge \omega + \tilde{L}\, d_{\tilde{V}}\omega = \sum_{i=0}^{n} \frac{\partial \tilde{L}}{\partial \kappa_i}\,(d_{\tilde{V}}\kappa_i) \wedge \omega + \tilde{L}\, d_{\tilde{V}}\omega \equiv (\mathrm{mod}\ d_{\tilde{H}})$$

$$\boxed{(d_{\tilde{V}}\kappa_i) \wedge \omega = (d_{\tilde{V}}\mathcal{D}\kappa_{i-1}) \wedge \omega = (d_{\tilde{V}} d_{\tilde{H}}\kappa_{i-1}) - \kappa_i d_{\tilde{V}}\omega = -\,(d_{\tilde{H}} d_{\tilde{V}}\kappa_{i-1}) - \kappa_i d_{\tilde{V}}\omega}$$

$$\begin{aligned} &\equiv \quad \textstyle\sum_{i=0}^{n} d_{\tilde{H}}\left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right)(d_{\tilde{V}}\kappa_{i-1}) - \left(\sum_{i=0}^{n}\frac{\partial \tilde{L}}{\partial \kappa_i}\kappa_i - \tilde{L}\right) d_{\tilde{V}}\omega \\ &= \quad \textstyle\sum_{i=0}^{n} (-\mathcal{D})\left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right)(d_{\tilde{V}}\kappa_{i-1}) \wedge \omega - (\ldots)\, d_{\tilde{V}}\omega. \end{aligned}$$

repeat!

Symbolic: recursively implement integration by parts, **not** the formulas.

$$\boxed{d_{\tilde{V}}\tilde{\lambda} \equiv \mathcal{E}\left(\tilde{L}\right) d_{\tilde{V}}\kappa \wedge \omega - \mathcal{H}\left(\tilde{L}\right) d_{\tilde{V}}\omega.}$$

$$\mathcal{E}\left(\tilde{L}\right) = \sum_{i=0}^{n} (-\mathcal{D})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}\left(\tilde{L}\right) = \sum_{i>j\geq 0}^{n} \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$$\lambda = L(x, u, u_1, \ldots, u_m)\, dx \quad \leftrightarrow \quad \tilde{\lambda} = \tilde{L}\left(\kappa, \kappa_1, \ldots, \kappa_n\right)\omega$$

$$d_V(dx) = 0 \qquad d_{\tilde{V}}(\omega) = \mathcal{B}(\vartheta_0) \wedge \omega$$

$$d_V(u) = \theta_0 \qquad d_{\tilde{V}}(\kappa) = \mathcal{A}(\vartheta_0)$$

$$\boxed{d_{\tilde{V}}\tilde{\lambda} \equiv \left[\mathcal{A}^*\mathcal{E}\left(\tilde{L}\right) - B^*\mathcal{H}\left(\tilde{L}\right)\right]\vartheta_0 \wedge \omega.}$$

Examples:

- **Euclidean group:** $SE(2) = SO(2) \ltimes R^2$

  $\mathcal{A}^* = \left(\frac{d}{ds}\right)^2 + \kappa^2$

  $\mathcal{B}^* = -\kappa$

- **Affine group:** $SA(2) = SL(2) \ltimes R^2$

  $\mathcal{A}^* = \left(\frac{d}{ds}\right)^4 + \frac{5}{3}\kappa\left(\frac{d}{ds}\right)^2 + \frac{5}{3}\kappa_s\left(\frac{d}{ds}\right) + \frac{1}{3}\kappa_{ss} + \frac{4}{9}\kappa^2$

  $\mathcal{B}^* = \frac{1}{3}\left(\frac{d}{ds}\right)^2 - \frac{2}{9}\kappa$

Example: Lagrangian invariant under $SE(2) = SO(2) \ltimes \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2\,(1+u_x^2)^{5/2}} dx = \int L dx \quad \Longleftrightarrow \quad \int \tfrac{1}{2}\kappa^2\, ds$$

$$\Big\downarrow E = \left(\tfrac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\tfrac{d}{dx}\right) \frac{\partial}{\partial u_x} \qquad\qquad \left(\left(\tfrac{d}{ds}\right)^2 + \kappa^2\right) \mathcal{E} + \kappa \mathcal{H} \Big\downarrow$$

$$E(L) = 0 \quad \Longleftrightarrow \quad \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

$E(L) =$

$$\frac{1}{2}\,\frac{30\,{u_2}^3\,{u_1}^2 - 5\,{u_2}^3 - 20\,u_2\,u_1\,u_3 - 20\,u_2\,{u_1}^3\,u_3 + 2\,u_4 + 4\,u_4\,{u_1}^2 + 2\,u_4\,{u_1}^4}{(1+{u_1}^2)^{(9/2)}}.$$

$(\ u_1 = u_x, \ldots, u_4 = u_{xxxx})$

Noether correspondence in the invariant bicomplex

Noether correspondence. Inductive approach.

Classical computation:

- find (generalized) infinitesimal symmetries of a Lagrangian (Lie's method),
- compute corresponding conservation laws.

Inductive approach

- reduce a Lagrangian by its group of point symmetries G,
- find (generalized) infinitesimal symmetries of the reduced Lagrangian (Lie's method),
- compute **corresponding** conservation laws (invariant with respect to G).

Infinitesimal symmetry condition for $\lambda \in \tilde{\Lambda}^{p,0}$.

$$\mathcal{L}_{prY}\lambda \equiv d_{\tilde{H}}\alpha \mod \tilde{\Lambda}^{p-1,1}$$

$$\Updownarrow$$

$$prY \lrcorner d_{\tilde{V}}\lambda = d_{\tilde{H}}\alpha, \text{ for } \alpha \in \tilde{\Lambda}^{p-1,0}.$$

Noether correspondence:

$$d_{\tilde{V}}\lambda - \partial_{\tilde{v}}\lambda = d_{\tilde{H}}\mu \text{ for } \mu \in \tilde{\Lambda}^{p-1,1} \Longrightarrow d_{\tilde{H}}P = 0 \text{ mod} EL, \text{ where}$$

$$\boxed{\pi = prY \lrcorner \mu + \alpha} \text{ is a conservation law.}$$

- $SE(2) \curvearrowright \mathbb{R}^2$-inv. Lagrangian: $\boxed{\lambda = \frac{1}{2}\kappa^2\omega}$, $\omega = ds + \frac{u_1}{\sqrt{1+u_x^2}}\theta$.

  $d_{\tilde{V}}\lambda = EL\,\vartheta \wedge \omega \; + d_H M$, where $\boxed{M = \kappa_1\vartheta_1 - \kappa\vartheta_2}$.

- Euler-Lagrange equation: $\boxed{EL = \kappa_{ss} + \frac{1}{2}\kappa^3.}$

- $Y = \xi(\kappa, \kappa_s)V$. symmetry condition $\Longrightarrow$ $\xi = \kappa_1\, f(\kappa^2 + 4\kappa_1^2)$.
  Take $Y = \kappa_1 V$

$$prY \lrcorner\, d_{\tilde{V}}\,\lambda = d_{\tilde{H}}A \text{ where } A = \kappa\kappa_2 + \frac{1}{4}\kappa^4 - \kappa_1^2.$$

- Conservation laws: $\boxed{P = Y \lrcorner M + A}$

$$P = \kappa_s^2 + \frac{1}{4}\kappa^4, \text{ with } \frac{d}{ds}P = \kappa_s(\kappa_{ss} + \frac{1}{2}\kappa^3) \equiv 0 \bmod EL.$$

- $SA(2) \curvearrowright \mathbb{R}^2$-inv. Lagrangian: $\boxed{\lambda = \mu\,\omega}$, $\mu = \frac{u_2 u_4 - \frac{5}{3} u_3^2}{u_2^{8}/3}$,

  $\omega = u_2^{1/3}\, dx + \frac{u_3}{3\, u_2^{5/3}} \theta$.

  $d_{\tilde{V}} \lambda = EL\, \vartheta \wedge \omega \; + d_H M$, where $\boxed{M = \frac{2}{3}\vartheta_1 - \frac{2}{3}\vartheta_2 - \vartheta_4}$.

- Euler-Lagrange equation: $\boxed{-(\frac{2}{3}\mu_2 + \frac{2}{9}\mu^2).}$

- $Y = \xi(\kappa, \kappa_s) V$. symmetry condition $\Longrightarrow \xi = \kappa_1\, f(\frac{2}{9}\kappa^3 + \kappa_1^2)$.

  Take $Y = \kappa_1 V$

$$prY \lrcorner\, d_{\tilde{V}}\, \lambda = d_{\tilde{H}} A \text{ where } A = \mu_4 + 2\mu\mu_2 + \frac{2}{27}\mu^3.$$

- Conservation laws: $\boxed{P = Y \lrcorner M + A}$

$$P = \frac{2}{27}\mu^3 + \frac{1}{3}\mu_1^2$$

Implemented relative to invariant frames

iVBStructure and iVarCalc programmed in Vessiot environment.
(*for maple worksheet contact iakogan@ncsu.edu*)

- integration by parts $\Rightarrow$ E.L equations, Helmholtz conditions.
- vector field prolongation
- exactness of interior rows of variational bicomplex
- Noether correspondence (non divergence symmetries)

Thank you!