Rational and Replacement Invariants
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Group actions are ubiquitous in mathematics. They arise in diverse areas of applications, from mechanics to computer vision. A classical but central problem is to compute a complete set of invariants.

Definitions

**Algebraic Group**

\( G \subset k^d \) is an algebraic variety. \( G \subset k[\lambda_1, \ldots, \lambda_n] \) is the ideal of \( G \)

\[ m: G \times G \to G \quad \text{and} \quad \pi: G \to G \]

\( \lambda^* : G \times k \to k^d \)

\( (\lambda, \mu) \mapsto \lambda \cdot \mu \)

\( \lambda^* : G \to G \)

\( \lambda \mapsto \lambda = \lambda \cdot 1 \)

\( \lambda^* : G \times k \to k^d \)

\( \lambda \cdot \lambda' = \alpha \quad \lambda = \lambda(\alpha) \)

\( \lambda^* : G \times k \to k^d \)

\( \lambda \mapsto \lambda = \lambda(\alpha) \quad \alpha \in k \)

**Rational Action on \( Z^d \)**

\( G \times Z \to Z \quad \lambda : (\mu, z) \to \lambda \cdot (\mu, z) \quad \lambda \cdot (\mu, z) = (\lambda^* \mu, \lambda^* z) \quad e = e \in G \)

**Field of Rational Invariants**

\( k[G] \)

**Rational Invariant**

\[ p(z) \in k[G] \quad p(z) \mapsto p(z) \quad \text{mod } G \]

**Finiteness**

\( k[G] \subset (k[r_1, \ldots, r_d]) \subset k[Z] \)

**Orbit separation**

\[ p(z) \in k[G] \quad p(z) \mapsto p(z) \quad \text{mod } G \]

**Examples**

**Graph of the action**

\[ O = \{ (z, \lambda) \in Z \times Z \mid 3 \in G \} \quad \text{s.t.} \quad \lambda \cdot z = z \]

It's ideal \( O = (G + (Z - z)) \times k[Z] \)

\( (Z - z) = (h_2 Z - g_2, g_1 \leq i \leq n) : h_2 \]

**Orbit extension of \( O \) in \( k[Z] \)**

**Invariance**

\[ \{ (z, \lambda) \in O \mid p(z, \lambda \cdot z) = p(z, \lambda \cdot z) \} \]

**The reduced Gröbner basis of \( O \)**

\[ \{ r_1, \ldots, r_d \} \] its coefficients

**Rewriting Algorithm**

\[ \tilde{Q} \]

In:

\[ Q, r \in k[Z] \]

**Output:**

\[ R \]

**Scalar**, translation, reflection, rotation.

**Cross-section of degree \( d \)**

A variety that intersects generic orbits in \( d \) simple points.

**Rational Invariants**

**Construction of Rational Invariants**

\[ \{ r_1, \ldots, r_d \} \text{ its coefficients} \]

**Replacing Invariants**

If \( d \) then \( \tilde{Q} \) is a reduced Gröbner basis

\[ \{ r_1, \ldots, r_d \} \]

**Equivalence of Curves \( \Rightarrow \) Object Recognition**

\[ Y \]

\[ X \]

**The Big Project: Invariant Differential Systems**

A Problem in Differential Elimination: Equations for \( \mathcal{O} \)

\[ \{ r_1, \ldots, r_d \} \]

**Note:**

**Fundamental invariants**

**Syzygies**

**Differential Rewriting of \( \mathcal{O} \)**

in terms of \( Z = (x_1, x_2, \ldots, x_{d+1}, y_1, \ldots, y_d) \) and \( \Delta = \{ \delta_1, \ldots, \delta_d \} \)

**Invariant Differential Invariants**

**Tips & Techniques**

The cross-section is chosen so that \( s_1, s_2, \ldots, s_d \) and \( \delta_1, \delta_2, \ldots, \delta_d \) depend only on \( s \) and its derivatives.

To simplify the commutation rules we chose

\[ Z = x_1, x_2, \ldots, x_{d+1}, y_1, \ldots, y_d \]

**Algebra of Differential Invariants**

Non-commutation of the derivations:

**Syzygies of the fundamental invariants:**

**Differential Invariant Projection**

Treat \( Z \cup Z^d \) in the differential polynomial ring with non trivial commutation rules [Hubert 05]

**Tips**

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