Research Overview of Irina Kogan

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My research area is the geometric study of differential and algebraic equations. I am especially interested in problems that can lead to efficient algorithms for either obtaining explicit solutions, or obtaining useful geometric information about the solution set of the equations. My main focus is on the problems of equivalence and symmetry under group actions. Such problems arise in various areas of mathematics, including differential and algebraic geometry, differential equations and mathematical physics. Their solution involves computation of invariants, that is the quantities that are unaffected by a group action.

I worked on algorithms for computing algebraic and differential invariants, equivalence problems for polynomial equations, symmetry reduction of variational problems, and applications of geometric invariants in computer vision. While maintaining my efforts in the above areas, I have recently extended my research to geometric study of hyperbolic conservation laws. This is the first of my projects that does not directly involve transformation groups and invariants, but relies on a general theory of exterior differential systems, and, in particular, on Cartan-Kähler theory. My past, current, and planned research projects are described in more details in the next paragraphs. My publications and preprints can be downloaded from my web-site `www.math.ncsu.edu/∼iakogan`.

Algebraic and Differential Invariants I became interested in algebraic invariant theory and its applications as an undergraduate student at Moscow Institute for Petrochemical and Natural Gas Industry. In my diploma project, conducted under the supervision of Nikolay Osetinsky, I used algebraic and combinatorial methods presented in [59] and [17] to develop an algorithm for computing the dimension of a space of polynomial invariants, of a given degree, for vectors in $\mathbb{R}^n$ and rank 2 tensors under the standard action of the orthogonal group O_n.

In my graduate work at the University of Minnesota I expanded my interest to differential invariants and related topics in differential geometry and algebra. In particular, I became interested in Cartan's solution of various equivalence problems based on the moving frame method and its modern more rigorous formulation [28], [38], [27], [15], [25], [56]. I became familiar with, at that time, newly developed generalization of the moving frame method by Fels and Olver [19]. This generalization leads to a invariantization process for functions, forms and differential operators, and is applicable to a wide range of group actions.

The computational difficulties of applying Fels-Olver method are often substantial, however, for groups of relatively large dimension. As part of my dissertation work [41], performed under Peter Olver's supervision, I developed two variations of the Fels-Olver algorithm: recursive and inductive constructions, that allow one to construct moving frame and invariants for the entire group in terms of the moving frame and invariants of its subgroups. This work gave rise to two publications [42] and [43]. I used the recursive approach to derive a separating set of differential invariants for homogeneous polynomials in three variables undergoing linear changes of variables [41], and the inductive approach to derive separating set of integral invariants for spatial curves [21] undergoing the affine transformations. More details on these projects are presented below. The inductive and recursive approach was also used by Smirnov, Yue and Horwood [62], [65], [32] to compute invariants and covariants of Killing tensors under the action of isometry groups.

In recent joint papers with E. Hubert [34] and [35] we obtained a purely algebraic formulation of the moving frame method for constructing local smooth invariants on a manifold under an action of a Lie group. This formulation gives rise to new, computationally effective algebraic algorithms for constructing rational and replacement invariants. These algorithms can be also used to obtain a generating set of differential invariants.

Symmetry and Equivalence of Polynomials Given two polynomials the goal is to decide whether one polynomial can be transformed to the other by a linear change of variables. If the answer is positive we call such polynomials equivalent. A more difficult problem is to obtain a complete classification of the equivalence classes, and to identify in each class a "simple" canonical representative (precise definition of "simple" is part of the problem). A related problem is that of finding the symmetry group of a given polynomial, i.e., all linear changes of variables that preserve a given polynomial.

These problems were actively studied by a variety of methods since XIX century (an overview can be found in [26], [30], [57], [54], [63], [33], [16]) Nevertheless, the complete solution is not known with the exception of the cases of polynomials in two or three variables of low degree. In [57] Olver proposed to address these problems via a geometric method initially developed by Cartan to solve equivalence problem for submanifolds.

In a joint paper with P. Olver [6] we applied Cartan's moving frame method to describe the symmetry groups of homogeneous polynomials in three variables (binary forms). We presented an algorithm, implemented in MAPLE, that determines the dimension of the symmetry group of a given binary form and, in the case when the symmetry group is finite, computes the symmetry group explicitly. We used classical results from group theory [8] to classify all possible groups of discrete symmetries. The computation becomes more challenging in the case of three variables. In [41] I have constructed, for the first time, a complete separating set of differential invariants for homogeneous polynomials in three variables (ternary forms), sufficient to solve the equivalence problem. In particular, I found a necessary and sufficient conditions for a ternary form to be equivalent to $x^n + y^n + z^n$. In my joint work with M. Moreno Maza [46] we used these invariants to conduct a careful study of the equivalence classes of ternary cubics under general complex linear changes of variables. Whereas the classification of cubics is known in both complex and real cases [7], [49], we provided a computationally efficient algorithm that determines the canonical form of an arbitrary cubic and explicitly finds corresponding linear changes of coordinates. We also classified the symmetry groups of ternary cubics.

Computation for polynomials in larger number of variables presently remains out of reach. Solution of the equivalence problem for systems of polynomials is another challenging extension. The newly obtained algebraic formulation of Cartan's moving frame method by E. Hubert and myself [34], [35] is better adapted to the algebraic nature of the problem than previous more geometric approach. Its application to the equivalence and symmetry problem of polynomials is one of my futures research projects. Since the solution sets of equivalent polynomials are related by a linear change of variables, such classification can facilitate finding the solution set of polynomial equations.

Symmetry Reduction of Variational Problems Many systems of differential equations or variational problems arising in geometry and physics admit natural groups of symmetries, that is, a group which maps a solution to another solution. With singular exceptions, symmetric variational functionals can be written in terms of differential invariants and invariant differential forms, and, therefore, reduced by a group of symmetries.

In joint papers with P. Olver [47], [48] we derived, for the first time, an explicit general formula for the invariant Euler-Lagrange operator applicable to variational problems in any number of independent and dependent variables. Invariant Euler-Lagrange operator, applied to a symmetry-reduced variational functional, directly produces symmetry-reduced Euler-Lagrange equations. Previously such operator was known only in a special case of first-order variational problems with one independent variable [14] and for some particular geometric examples [29], [2].

In order to obtain a formula for the invariant Euler-Lagrange operator we performed variational calculus relative to an invariant basis of first-order linear differential operators (invariant frame) and an invariant basis of differential one-forms (invariant coframe). I am now using the same machinery to explore Noether's correspondence for the reduced variational problems [44].

I implemented many of the algorithms for computations relative to invariant frames and coframes in a MAPLE package iVB (invariant Variational Bicomplex). It includes, in particular, computation of the structure equations for an invariant frame and coframe, recurrence formulae, integration-by-parts, prolongation of vector fields in invariant frames, and invariant Euler-Lagrange operators. This package uses a package VESSIOT [3] developed by Ian Anderson to perform a wide range of symbolic computations in differential geometry. Recently Anderson extended VESSIOT package to an even more powerful suite of programs DIFFERENTIAL GEOMETRY [1] included in MAPLE 11 release. In collaboration with my graduate student Joseph Burdis, we are currently improving and adapting iVB package [45] to be compatible with the DIFFERENTIAL GEOMETRY package, and therefore executable within standard releases of MAPLE. We also preparing a publication that contains a detailed justification and description of algorithm for performing differential and variational calculus in invariant frames [12].

Geometric Invariants in Computer Vision Invariants under the actions of the Euclidean, affine and projective groups are widely used in image processing and computer vision (see, for instance, [55], [64], [18], [13].) Differential invariants, such as Euclidean curvature and torsion for space curves, are the most classical. The affine and projective counterparts of curvature and torsion may also be defined. The practical utilization of differential invariants is, however, limited due to their high sensitivity to noise.

Since integration reduces the effect of noise, integral invariants hold a clear advantage in practical applications. Until a recent work by S. Feng, H. Krim and myself [21], the explicit expressions for integral invariants appear to be known only for planar curves [60], [50], [51], [31], and remained elusive for curves in 3D, primarily due to computational complexity of their derivation. An inductive variation of the moving frame method [43], allowed us to overcome computational difficulties, and to derive affine integral invariants in terms of Euclidean integral invariants. In [22] we used these integral invariants to define signatures, that classify curves up to Euclidean and affine transformations. We presented two types of signatures, the global signature and the local signature. Both signatures are independent of parameterization (curve sampling). The global signature depends on the choice of the initial point and does not allow one to compare fragments of curves, and is therefore sensitive to occlusions. The local signature, although slightly more sensitive to noise, is independent of the choice of the initial point and is not sensitive to occlusions in an image. It can be used to establish local equivalence of curves. The robustness of these invariants and signatures in their application to the problem of classification of noisy spatial curves extracted from a 3D object was analysed and is promising. We made an initial investigation of an application of integral invariants to face recognition in [23]. I plan to continue an exploration of applied and theoretical questions related to integral invariants.

Geometric Study of Hyperbolic Conservation Laws In a joint work with H. K. Jenssen [39],[40] we address the problem of constructing hyperbolic conservation laws with prescribed eigencurves. We consider a system of n conservation laws in one space-dimension x and one time-dimension t, written in canonical form:

$$u_t + f(u)_x = 0\,.$$

Here the unknown state $u = u(t, x) \in \mathbb{R}^n$ is assumed to vary over some open subset $\Omega \subset \mathbb{R}^n$ and the flux f is a nonlinear map from Ω into $\mathbb{R}^n$. The eigenvalues and eigenvectors of the Jacobian matrix $Df(u)$ provide information that is used to solve the Cauchy problem for this system. If eigenvalues are strictly increasing, $\lambda_1(u) < \cdots < \lambda_n(u), \forall u \in \Omega$, the system of conservation laws is called strictly hyperbolic. The geometric properties of the integral curves of eigenvector fields of Df play a key role. Together with the so-called Hugoniot curves these form *wave curves* that are used to build solutions. In searching for systems whose wave curves have special properties one is naturally lead to ask what freedom one has in *prescribing* such curves. A complete answer to this question becomes rather subtle for $n > 2$. We utilize a geometric approach for to this problem in the spirit of [61], and the theory of exterior differential forms including Cartan-Kahler theory as described, for instance, in [11] and [37]. We note, however, that despite the similarity of the terminology our problem differs from the one considered in [9] and [10].

Future Research I am currently working on the completion of four research papers: "Algorithmic approach to differential and variational calculus in invariant frames" [12], joint with my Ph. D. student Joseph Burdis; "Noether's correspondence in the invariant variational complex" [44]; "Systems of hyperbolic conservation laws with prescribed eigencurves" [40] with Kris Jenssen; and "Face representation and recognition using Euclidean-invariant integral signatures" [24] with Shuo Feng and Hamid Krim. Each of these projects can be continued in a number of interesting directions.

I would like to explore applicability of the algorithms for computations in invariant frames developed in [12] to the problems of symmetry reduction of differential equations and exterior differential systems (see recent works of Anderson and Fels [4], [20]), as well as the extension of the algorithms to the symmetry reduction under the actions of Lie pseudo-groups explored by Olver and Pohjanpelto [58]. Consideration of the invariant inverse problem for calculus of variations [5] is a natural continuation of the invariant Noether's correspondence project

[44]. Extensions to discrete variational calculus in line of works by Marsden and West [53], Hydon and Mansfield [36], and Mansfield [52] is of interest.

The continuation of integral invariants project includes computation of integral invariants for various geometric groups acting on $\mathbb{R}^2$ and $\mathbb{R}^3$ (only few cases are explicitly computed at present); robust invariant numerical approximation of the integral signature construction; theoretical results about the structure of the set of integral invariants; and algorithms that use integral signature construction to extract information about the symmetries of the object. In the near future I plan to pursue these problems further. My Ph. D. student, Kathleen Iwancio Thompson, is currently involved in this project.

Jointly with Kris Jenssen we will continue the exploration of the geometric properties of the systems of conservation laws [40], in particular geometric properties of Hugoniot curves, and the geometric conditions for the existence of entropies.

I remain interested in both algebraic and differential aspects of computational invariant theory. In particular, I want to explore applications of the algebraic formulation of the moving frame method to the solution of equivalence problems for polynomials.

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