# Invariant Variational Problems
# &
# Invariant Curve Flows

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North Carolina, April, 2009

# Basic Notation

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$u^\alpha_J = \partial_J u^\alpha$ — partial derivatives

$F(x, u^{(n)}) = F(\ \ldots\ x^k\ \ldots\ u^\alpha_J\ \ldots\ )$ — differential function

$G$ — transformation group acting on the space of independent and dependent variables

# Variational Problems

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations: $\mathbf{E}(L) = 0$

components: $$\mathbf{E}_\alpha(L) = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J}$$

$$D_k F = \frac{\partial F}{\partial x^k} + \sum_{\alpha, J} u^\alpha_{J,k} \frac{\partial F}{\partial u^\alpha_J}$$

— total derivative of $F$ with respect to $x^k$

# Invariant Variational Problems

According to Lie, any $G$–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is $G$-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P( \ \ldots \ \mathcal{D}_K I^\alpha \ \ldots \ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit $G$ as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F( \ \ldots \ \mathcal{D}_K I^\alpha \ \ldots \ ) = 0$$

## Main Problem:

Construct $F$ directly from $P$.

(*P. Griffiths, I. Anderson* )

# Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$ — curvature (differential invariant)

$$ds = \sqrt{1+u_x^2}\,dx$$ — arc length

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$ — arc length derivative

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Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x,u^{(n)})\,dx = \int P(\kappa,\kappa_s,\kappa_{ss},\ \ldots\ )\,ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa,\kappa_s,\kappa_{ss},\ \ldots\ ) = 0$$

# Euclidean Curve Examples

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Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1+u_x^2}\; dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

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The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1+u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

## General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

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## Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

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## Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\ (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

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$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P) = 0}$$

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The Elastica: $\mathcal{I}[u] = \int \frac{1}{2}\,\kappa^2\, ds \qquad P = \frac{1}{2}\,\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\frac{1}{2}\,\kappa^2$$

$$\boxed{\begin{aligned}\mathbf{E}(L) &= (\mathcal{D}^2 + \kappa^2)\,\kappa + \kappa\,(\,-\,\tfrac{1}{2}\,\kappa^2\,)\\ &= \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0\end{aligned}}$$

# The shape of a Möbius strip

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**The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].**

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

**Figure 1 Photo of a paper Möbius strip of aspect ratio $2\pi$.** The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

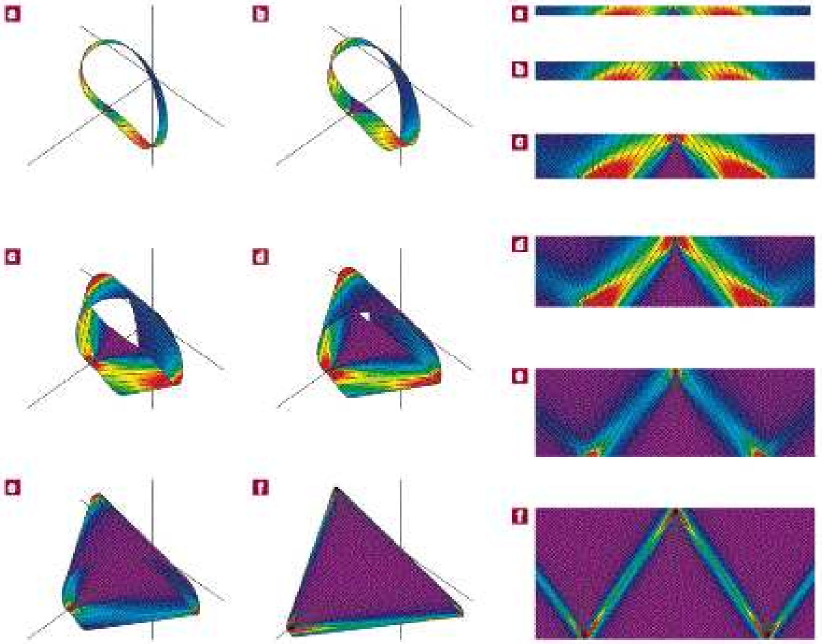

**Figure 2 Computed Möbius strips.** The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

# Moving Frames

$G$ — $r$-dimensional Lie group acting on $M$

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — $n^{\text{th}}$ order jet bundle for
$p$-dimensional submanifolds $N = \{\, u = f(x) \,\} \subset M$

$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$ — coordinates on $\mathrm{J}^n$

$G$ acts on $\mathrm{J}^n$ by prolongation (chain rule)

**Definition.**

An $n^{\text{th}}$ order *moving frame* is a $G$-equivariant map

$$\rho = \rho^{(n)} : V \subset \mathrm{J}^n \longrightarrow G$$

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Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note: $\rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

**Theorem.** A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $G$ acts freely and regularly near $z^{(n)}$.

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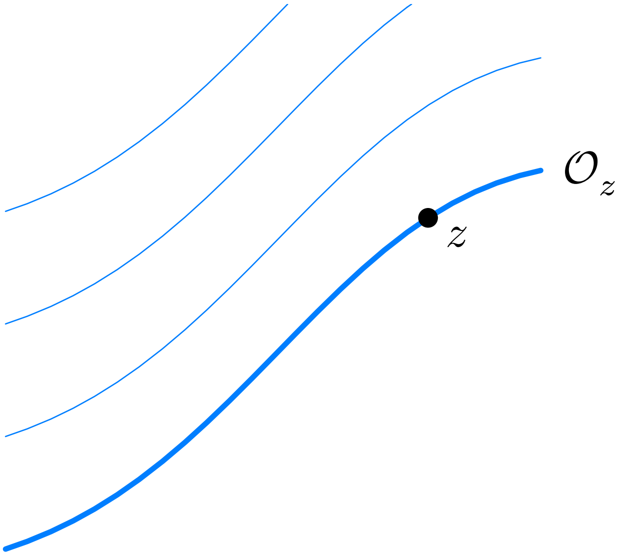
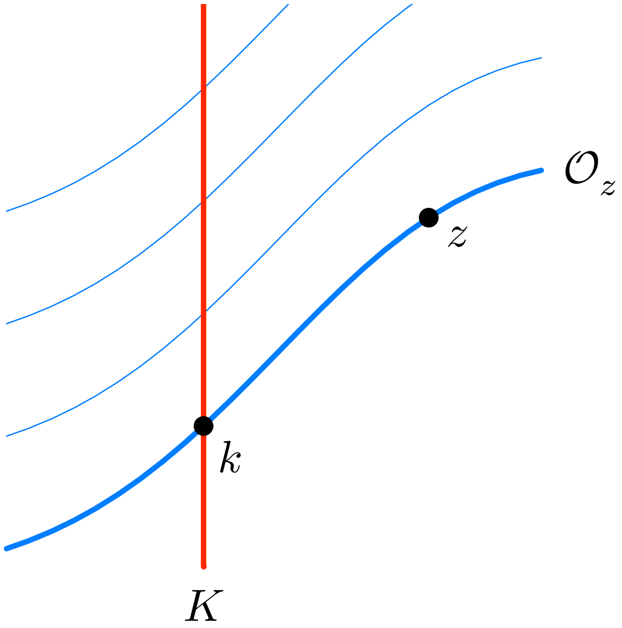
- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $g \cdot z = z$ if and only if $g = e$.
- locally free — the orbits have the same dimension as $G$.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  ( $\not\approx$ irrational flow on the torus)

# Geometric Construction



$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$

# Geometric Construction



Normalization = choice of cross-section to the group orbits

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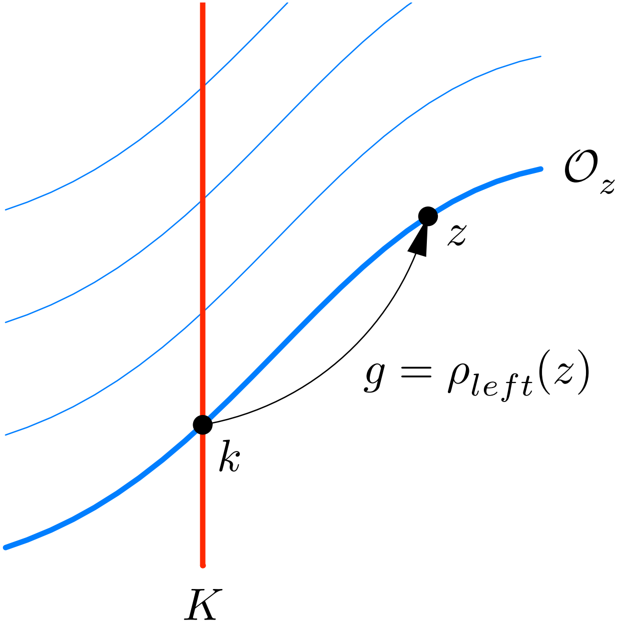
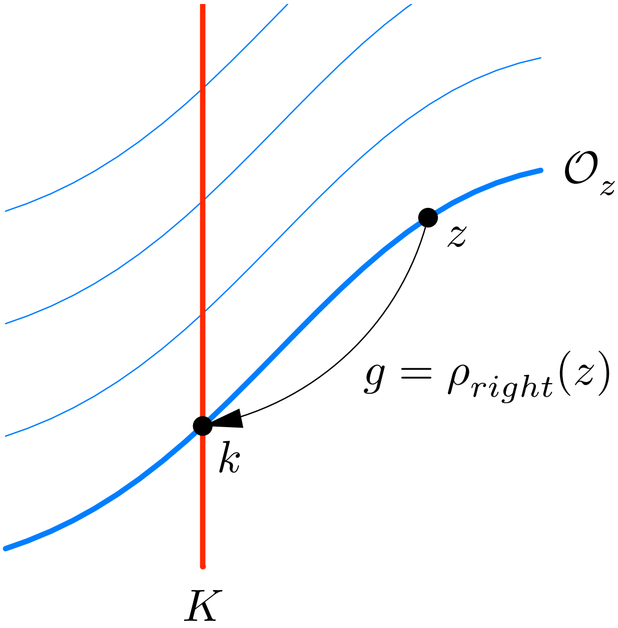


Normalization = choice of cross-section to the group orbits

# Geometric Construction



Normalization = choice of cross-section to the group orbits

# The Normalization Construction

**1.** Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$$\Longrightarrow \quad \textit{Implicit differentiation}$$

**2.** From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations*:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

**3.** Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

**4.** Invariantization: substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants:

$$I^{(n)}(x, u^{(n)}) = \iota(z^{(n)}) = w^{(n)}(\rho(z^{(n)}), z^{(n)}))$$

# Euclidean plane curves $\qquad G = \mathrm{SE}(2)$

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Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

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Write out the group transformations

$$\left.\begin{aligned} y &= x\cos\phi - u\sin\phi + a \\ v &= x\cos\phi + u\sin\phi + b \end{aligned}\right\} \qquad w = R\,z + c$$

## Prolong to $\mathrm{J}^n$ via implicit differentiation

$$y = x\cos\phi - u\sin\phi + a \qquad v = x\cos\phi + u\sin\phi + b$$

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi} \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi - u_x\sin\phi)\,u_{xxx} - 3\,u_{xx}^2\sin\phi}{(\cos\phi - u_x\sin\phi)^5}$$

$$\vdots$$

Choose a cross-section, or, equivalently a set of $r = \dim G = 3$ normalization equations:

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

Solve the normalization equations for the group parameters:

$$\phi = -\tan^{-1} u_x \qquad a = -\frac{x + uu_x}{\sqrt{1+u_x^2}} \qquad b = \frac{xu_x - u}{\sqrt{1+u_x^2}}$$

The result is the right moving frame $\rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

Substitute into the moving frame formulas for the group parameters into the remaining prolonged transformation formulae to produce the basic differential invariants:

$$\begin{aligned} v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\ v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3\,u_x u_{xx}^2}{(1+u_x^2)^3} \\ v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3 &&= \cdots \end{aligned}$$

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**Theorem.** All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots$$

The invariant differential operators and invariant differential forms are also substituting the moving frame formulas for the group parameters:

Invariant one-form — arc length

$$dy = (\cos\phi - u_x \sin\phi)\, dx \quad \longmapsto \quad ds = \sqrt{1+u_x^2}\; dx$$

Invariant differential operator — arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi - u_x\sin\phi}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

# Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization.

The invariantization $I = \iota(F)$ is the unique invariant function that agrees with $F$ on the cross-section: $I \mid K = F \mid K$.

Invariantization respects algebraic operations, and provides a canonical projection that maps objects to their invariantized counterparts.

$$\iota : \quad \begin{array}{ccc} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \\ \begin{array}{c}\text{Differential} \\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential} \\ \text{Operators}\end{array} \end{array}$$

## Fundamental differential invariants = invariantized jet coordinates

$$\boxed{H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(l)}) = \iota(u^\alpha_K)}$$

The constant differential invariants, coming from the moving frame normalizations, are known as the *phantom invariants*. The remaining non-constant differential invariants are the *basic invariants* and form a complete system of functionally independent differential invariants.

**Invariantization of differential functions:**

$$\iota\,[\,F(\,\ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\,)\,] = F(\,\ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\,)$$

**Replacement Theorem:**

If $J$ is a differential invariant, then $\iota(J) = J$.

$$J(\,\ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\,) = J(\,\ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\,)$$

# The Infinite Jet Bundle

Jet bundles

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = ( \ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$\Longrightarrow$ Taylor series

# Differential Forms

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

- Horizontal one-forms

$$dx^1, \ldots, dx^p$$

- Contact (vertical) one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i$$

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Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A^\alpha_J\, \theta^\alpha_J$$

# The Variational Bicomplex

$\Longrightarrow$ *Dedecker, Vinogradov, Tsujishita, I. Anderson, ...*

Bigrading of the differential forms on $\mathrm{J}^\infty$:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

# Vertical and Horizontal Differentials

$$F(x, u^{(n)}) \qquad \text{— differential function}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— first variation}$$

$$d_H (dx^i) = d_V (dx^i) = 0,$$

$$d_H (\theta_J^\alpha) = \sum_{i=1}^{p} dx^i \wedge \theta_{J,i}^\alpha \qquad\qquad d_V (\theta_J^\alpha) = 0$$

# The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

$x$ — independent variable

$u$ — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x \, dx$$

$$\theta_x = du_x - u_{xx} \, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} \, dx$$

$$\vdots$$

$$\theta = du - u_x\,dx, \qquad \theta_x = du_x - u_{xx}\,dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\,dx$$

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Differential:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial u}\,du + \frac{\partial F}{\partial u_x}\,du_x + \frac{\partial F}{\partial u_{xx}}\,du_{xx} + \cdots \\ &= (D_x F)\,dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \cdots \\ &= d_H\,F + d_V\,F \end{aligned}$$

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Total derivative:

$$D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\,u_x + \frac{\partial F}{\partial u_x}\,u_{xx} + \frac{\partial F}{\partial u_{xx}}\,u_{xxx} + \cdots$$

# The Variational Bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & \Omega^{0,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,3} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^3 \arrow[u, "\delta"] \\
 & \Omega^{0,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,2} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^2 \arrow[u, "\delta"] \\
 & \Omega^{0,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,1} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^1 \arrow[u, "\delta"] \\
\mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,0} \arrow[u, "d_V"] \arrow[ur, "\mathbf{E}"'] &
\end{tikzcd}$$

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Lagrangians

# The Variational Bicomplex

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Lagrangians   PDEs (Euler–Lagrange)

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 & \Omega^{0,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,3} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^3 \arrow[u, "\delta"] \\
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 & \Omega^{0,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,1} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^1 \arrow[u, "\delta"] \\
\mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,0} \arrow[u, "d_V"] \arrow[ur, "\mathbf{E}"'] &
\end{tikzcd}$$

Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

# The Variational Bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & \Omega^{0,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,3} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^{3} \arrow[u, "\delta"] \\
 & \Omega^{0,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,2} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^{2} \arrow[u, "\delta"] \\
 & \Omega^{0,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,1} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^{1} \arrow[u, "\delta"] \\
\mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{p,0} \arrow[u, "d_V"] \arrow[ur, "\mathbf{E}"'] &
\end{tikzcd}$$

conservation laws   Lagrangians   PDEs (Euler–Lagrange)   Helmholtz conditions

# The Variational Derivative

$$\boxed{\mathbf{E} = \pi \circ d_V}$$

$d_V$ — first variation

$\pi$ — integration by parts = mod out by image of $d_H$

$$\begin{tikzcd}
\Omega^{p,0} \arrow[r, "d_V"] & \Omega^{p,1} \arrow[r, "\pi"] & \mathcal{F}^1 = \Omega^{p,1} / \, d_H \, \Omega^{p-1,1} \\
\lambda = L \, d\mathbf{x} \arrow[r] & \displaystyle\sum_{\alpha,J} \frac{\partial L}{\partial u^\alpha_J} \theta^\alpha_J \wedge \, d\mathbf{x} \arrow[r] & \displaystyle\sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L) \; \theta^\alpha \wedge \, d\mathbf{x} \\
\text{Variational problem} \arrow[r] & \text{First variation} \arrow[r] & \text{Euler–Lagrange source form}
\end{tikzcd}$$

# **The Simplest Example:** $(x,u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

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Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = \ d_V\, \lambda = \ d_V\, L \wedge dx$$

$$= \left( \frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge \ dx \ \in \ \Omega^{1,1}$$

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Integration by parts — compute modulo im $d_H$ :

$$d\lambda \ \sim \ \delta\lambda = \left( \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \ \in \ \mathcal{F}^1$$

$$= \mathbf{E}(L)\, \theta \wedge\, dx$$

$\Longrightarrow$ Euler-Lagrange source form.

To analyze invariant variational problems, invariant conservation laws, etc., we apply the moving frame invariantization process to the variational bicomplex:

# The Invariant Variational Complex

$\Longrightarrow$ Joint work with Irina Kogan.

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$\iota$ — invariantization associated with moving frame $\rho$.

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- Fundamental differential invariants

$$H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(n)}) = \iota(u^\alpha_K)$$

# The Invariant Variational Complex

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---

$\iota$ — invariantization associated with moving frame $\rho$.

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(n)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

# The Invariant “Quasi–Tricomplex”

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{aligned} d_{\mathcal{H}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2} \end{aligned}$$

---

Key fact: invariantization and differentiation *do not commute*:

$$d\,\iota(\Omega) \neq \iota(d\Omega)$$

# The Universal Recurrence Formula

$$\boxed{\boxed{d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]}}$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — invariantized dual Maurer–Cartan forms

$\Longrightarrow$ uniquely determined by the recurrence formulae for the phantom differential invariants

$$\boxed{\boxed{\; d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\left[\mathbf{v}_{\kappa}(\Omega)\right] \;}}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting $\Omega$ range over the basic functions and differential forms!

$$\boxed{\boxed{\; d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)] \;}}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Moreover, determining the structure of the differential invariant algebra and invariant variational bicomplex requires only linear differential algebra, and not any explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

# Euclidean plane curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(\, F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots\,)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots\;)$$

## Invariant arc length form

$$dy = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta$$

$$\begin{aligned} \varpi = \iota(dx) \quad &= \quad \omega \quad + \quad \eta \\ &= \quad \sqrt{1+u_x^2}\, dx \quad + \quad \frac{u_x}{\sqrt{1+u_x^2}}\, \theta \end{aligned}$$

$$\Longrightarrow \quad \theta = du - u_x\, dx$$

---

## Invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1+u_x^2}} \qquad\qquad \vartheta_1 = \iota(\theta_x) = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Basic recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\nu^1 + \iota(\mathbf{v}_2(F))\,\nu^2 + \iota(\mathbf{v}_3(F))\,\nu^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,$$

$$0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,$$

$$0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,$$

to solve for the Maurer–Cartan forms:

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

Recurrence formulae:

$$d\kappa = d\iota(u_{xx}) = \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3$$

$$= \iota(u_{xxx}\,dx + \theta_{xx}) - \iota(3\,u_x u_{xx})\,(\kappa\,\varpi + \vartheta_1) = I_3\varpi + \vartheta_2.$$

Therefore,

$$\mathcal{D}\kappa = \kappa_s = I_3, \qquad d_{\mathcal{V}}\,\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

where the final formula follows from the contact form recurrence formulae

$$d\vartheta = d\iota(\theta_x) = \varpi \wedge \vartheta_1, \qquad d\vartheta_1 = d\iota(\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta$$

which imply

$$\vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Similarly,

$$d\varpi = \iota(d^2x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx))$$

$$= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta.$$

In particular,

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

---

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta} \qquad \boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

# Plane Curves

Invariant Lagrangian:

$$\widetilde{\lambda} = L(x, u^{(n)})\, dx = P(\kappa, \kappa_s, \ldots)\, \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\sim\; \mathbf{E}(L)\, \vartheta \wedge \varpi$$

---

Invariant Integration by Parts Formula

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \;\sim\; -\, (\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

---

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &\sim\; \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + \mathcal{H}(P)\; d_{\mathcal{V}}\, \varpi \end{aligned}$$

## Vertical differentiation formulae

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta) \qquad\qquad \mathcal{A} \quad \text{— “Eulerian operator”}$$

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi \qquad\qquad \mathcal{B} \quad \text{— “Hamiltonian operator”}$$

---

$$\begin{aligned} d_{\mathcal{V}}\,\widetilde{\lambda} \;&\sim\; \mathcal{E}(P)\;\mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P)\;\mathcal{B}(\vartheta) \wedge \varpi \\ &\sim\; \Big[\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\,\Big]\;\vartheta \wedge \varpi \end{aligned}$$

---

## Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = 0}$$

# Euclidean Plane Curves

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Eulerian operator

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

---

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Hamiltonian operator

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

---

Euclidean–invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(P) + \kappa\,\mathcal{H}(P).$$

# Invariant Plane Curve Flows

$G$ — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

$G$–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- $I, J$ — differential invariants
- $\mathbf{t}$ — “unit tangent”
- $\mathbf{n}$ — “unit normal”

$\mathbf{t}$, $\mathbf{n}$ — basis of the invariant vector fields dual to the invariant one-forms:

$$\langle\, \mathbf{t}\,;\varpi\,\rangle = 1, \qquad \langle\, \mathbf{n}\,;\varpi\,\rangle = 0,$$

$$\langle\, \mathbf{t}\,;\vartheta\,\rangle = 0, \qquad \langle\, \mathbf{n}\,;\vartheta\,\rangle = 1.$$

---

$$C_t = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set $I$ to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:

# Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

**Examples — Euclidean–invariant curve flows**

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:

  $$C_t = \mathbf{n}_{\text{equi}-\text{affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

# Intrinsic Curve Flows

**Theorem.** The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$\mathcal{B}$ — invariant Hamiltonian operator

# Normal Evolution of Differential Invariants

**Theorem.** Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

---

Invariant variations:

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant linearization operator of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A}_\kappa + \kappa\,\kappa_s$ — invariant linearization operator of $\kappa_s$.

# Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

*Warning*: For non-intrinsic flows, $\partial_t$ and $\partial_s$ *do not commute*!

---

Grassfire flow: $J = 1$

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \ldots$$

$\Longrightarrow$ caustics

# Signature Curves

**Definition.** The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left( \kappa \ , \ \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

# Equivalence and Signature Curves

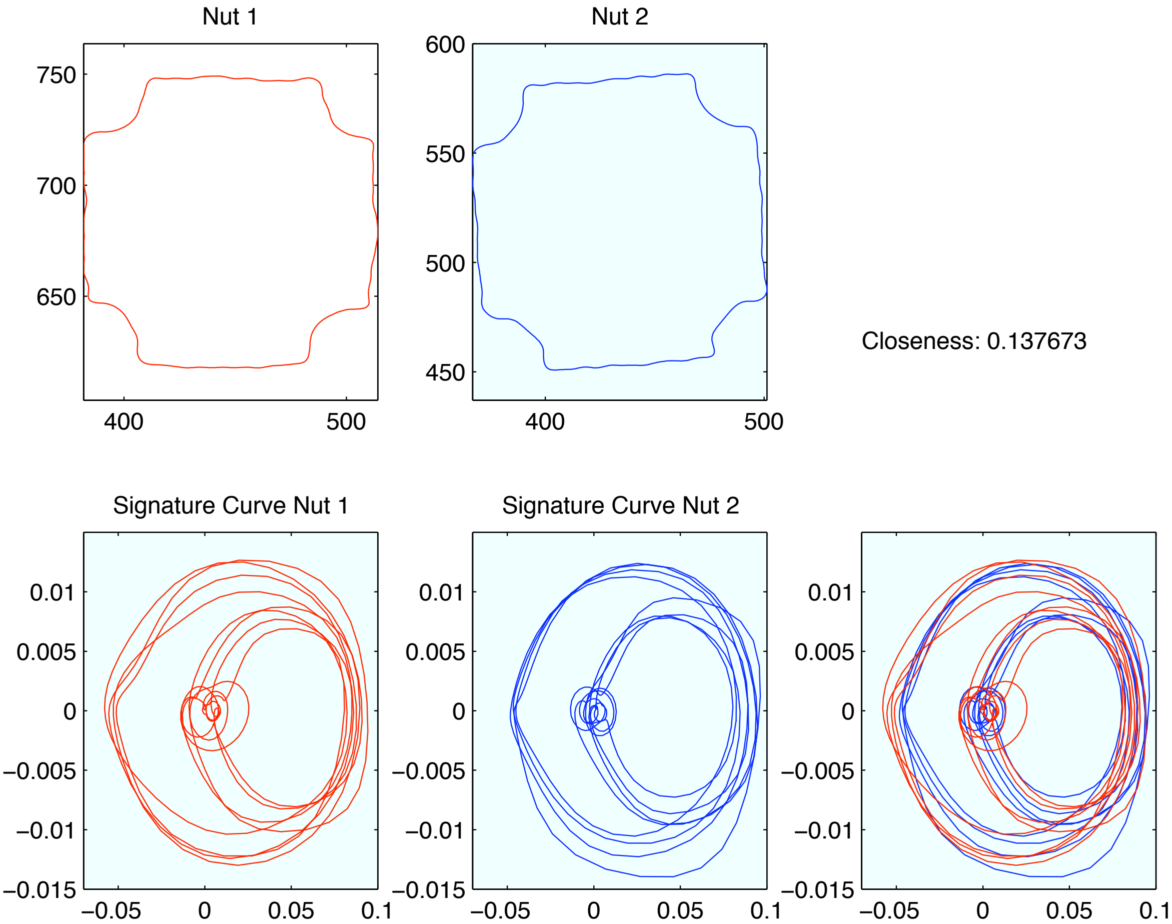
**Theorem.** Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

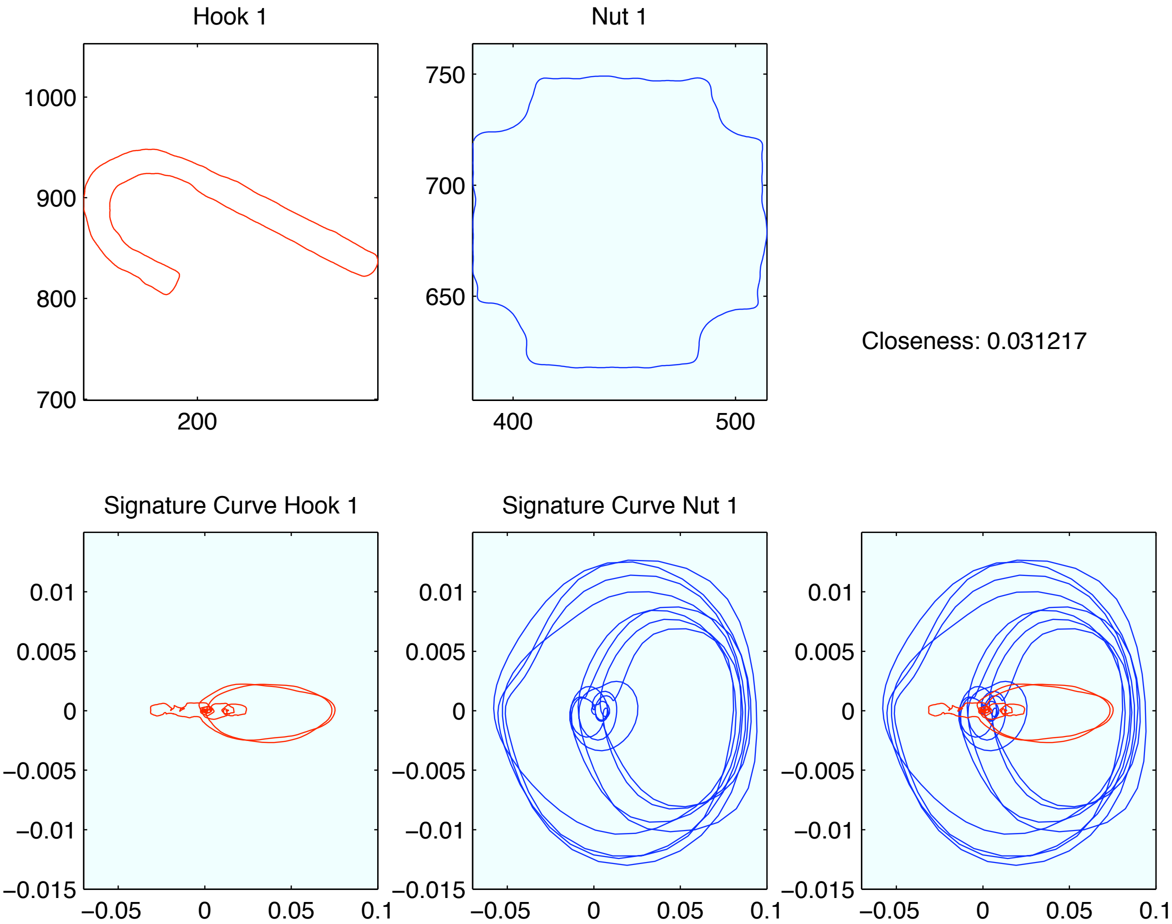
$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition


Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
1000
900
800
700
200
750
700
650
400
500
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

# Euclidean Signature Evolution

Evolution of the Euclidean signature curve

$$\kappa_s = \Phi(t, \kappa).$$

Grassfire flow:

$$\frac{\partial \Phi}{\partial t} = 3\,\kappa\,\Phi - \kappa^2\,\frac{\partial \Phi}{\partial \kappa}\ .$$

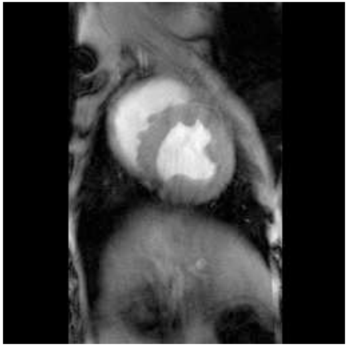
Curve shortening flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3 \Phi_{\kappa} + 4\,\kappa^2 \Phi.$$

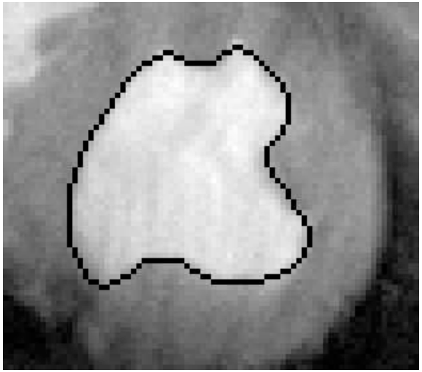
Modified Korteweg-deVries flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\,\Phi^2\,\Phi_{\kappa}\,\Phi_{\kappa\kappa} + 3\,\kappa\,\Phi^2.$$

# Canine Left Ventricle Signature

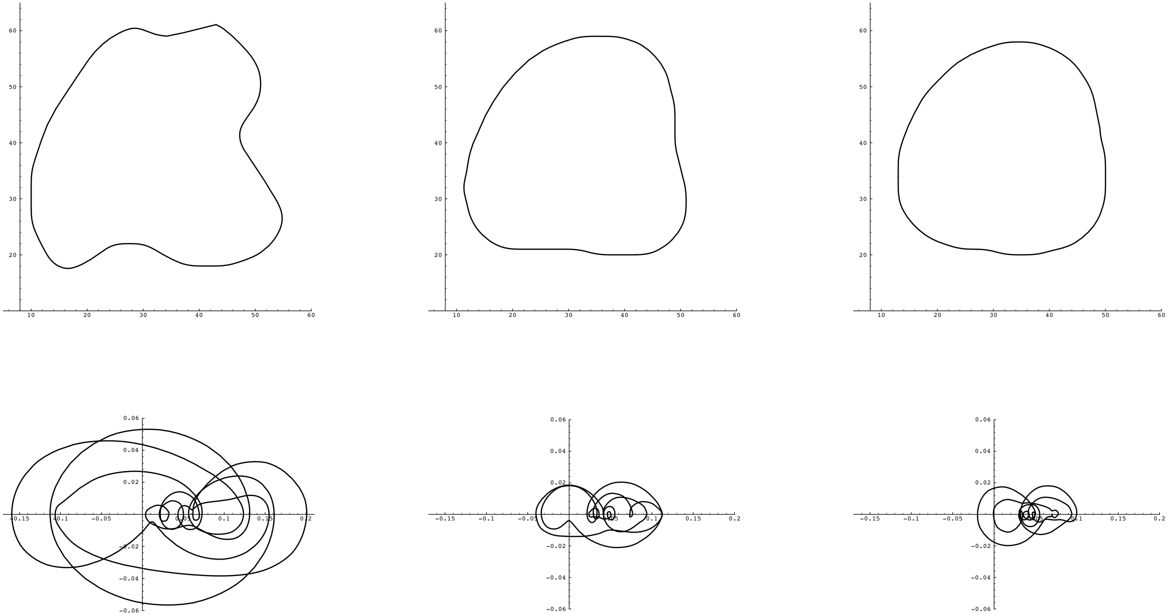


Original Canine Heart
MRI Image



Boundary of Left Ventricle

# Smoothed Ventricle Signature



# Intrinsic Evolution of Differential Invariants

**Theorem.**

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \qquad (*)$$

---

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, Anco, and many more ...

# Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta\wedge\varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s\mathcal{D}^{-1}\cdot\kappa$$

---

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\,\kappa^2\kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

# Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta), \qquad\qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2,$$

$$\mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa,$$

$$\begin{aligned}\mathcal{R} &= \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa\end{aligned}$$

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$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + 2\,\kappa\,\kappa_{ss} + \tfrac{4}{3}\,\kappa_s^2 + +\tfrac{5}{9}\,\kappa^2\,\kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

# Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \dfrac{2\tau}{\kappa} D_s^2 + \dfrac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \dfrac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \dfrac{1}{\kappa} D_s^3 - \dfrac{\kappa_s}{\kappa^2} D_s^2 + \dfrac{\kappa^2 - \tau^2}{\kappa} D_s + \dfrac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)