Projective Equivalence of Polynomial Subspaces

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Talk outline

- The equivalence problem for finite-dimensional polynomial subspaces

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- The Wronski map

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- Reduction to the equivalence problem for binary forms

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- Concluding remarks

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- $\mathcal{G}_k\mathcal{P}_n$: the variety of k-dimensional subspaces of $\mathcal{P}_n$

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The equivalence problem

For polynomial subspaces $U, \hat{U} \in \mathcal{G}_k\mathcal{P}_n$ with bases $p_1, \dots, p_k \in U$ and $\hat{p}_1, \dots, \hat{p}_k \in \hat{U}$, write $U \sim \hat{U}$ if there exist a fractional linear transformation and a $k \times k$ invertible matrix A_{ij} such that

$$\hat{p}_i(\hat{z}) = (c\hat{z}+d)^n \sum_{j=1}^{k} A_{ij}\, p_j\left(\frac{a\hat{z}+b}{c\hat{z}+d}\right), \quad i = 1, \dots, k.$$

Degree of the Wronskian $\mathrm{W}(p_1, \ldots, p_k)$

Proposition

For $p_1, \ldots, p_k \in \mathcal{P}_n$ and $\ell = n + 1 - k$ (the codimension),

$$\deg \mathrm{W}(p_1, \ldots, p_k) \leq k\ell$$

with equality iff $p_1, \ldots, p_k$ are linearly independent modulo $\mathcal{P}_{\ell-1}$.

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- Without loss of generality, $p_i(z) = z^{\nu_i} +$ lower degree terms, where $0 \leq \nu_1 < \cdots < \nu_k \leq n$ are the degree pivots.

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- Observe that $\nu_i = n - i$ iff $\lambda_1 = \cdots = \lambda_k = \ell$.
- Moreover, $\deg \mathrm{W}(p_1, \dots, p_k) = \deg \mathrm{W}(z^{\nu_1}, \dots, z^{\nu_k}) = \sum_{i=1}^{k} \lambda_i$.

The Plücker embedding

- Identify a subspace $U \in \mathcal{G}_k\mathcal{P}_n$ with primitive multi-vector span $p_1 \wedge \cdots \wedge p_k$ where $p_1, \ldots, p_k \in U$ is a basis.

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- The degree counts the standard Young tableaux with $\#$rows $\leq k$ and $\#$columns $\leq \ell$:

$$d(k,\ell) = \frac{(k-1)!!(\ell-1)!!}{n!!}(k\ell)!, \quad \text{where } j!! = 1!2!\cdots j!,$$

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- $\mathrm{PSL}_2 \mathbb{C}$ equivariance: for $U \in \mathcal{G}_k \mathcal{P}_n$, the projective Wronski map counts the intersections of the $(k-1)$-dimensional flat $[U]$ with the $(\ell - 1)$-dimensional developable variety generated by a $\mathrm{PSL}_2 \mathbb{C}$-invariant rational normal curve in $n = (k + \ell - 1)$-dimensional projective space $\mathcal{G}_1 \mathcal{P}_n$. Counting multiplicities, there are exactly $k\ell$ such intersections.

Geometric proof of equivariance

- Set
$$N(z) = (w-z)^k \wedge \cdots \wedge (w-z)^n \in \Lambda^k \mathcal{P}_n(w) \otimes \mathcal{P}_n(z).$$
For polynomials $p_i(w) = \sum_{j=0}^n p^{(j)}(z)(w-z)^j/j!$ we have
$$W(p_1, \ldots, p_k)(z)(1 \wedge w \wedge \cdots \wedge w^n) = (k-1)!!\, p_1(w) \wedge \cdots \wedge p_k(w) \wedge N(z)$$
- Regard $z \mapsto (w-z)^n$ as a realization of the rational normal curve in $\mathcal{G}_1\mathcal{P}_n(w)$. The 1-parameter family of corresponding $(\ell-1)$ dimensional osculating flats is $[N(z)] \subset \mathcal{G}_1\mathcal{P}_n$.
- Let $U \in \mathcal{G}_k\mathcal{P}_n$ with basis $p_1, \ldots, p_k$ By above formula, the roots of the Wronskian $W(U) = [\mathrm{W}(p_1, \ldots, p_k)]$ correspond to points on the rational normal curve where the $(\ell-1)$-dimensional flat $[N(z)]$ touches the $(k-1)$ dimensional flat U in the ambient $(k+\ell-1)$ dimensional projective space $\mathcal{G}_1\mathcal{P}_n$.
- The order of the root corresponds to the dimension of the intersection plus 1.
- Equivariance follows from $\mathrm{PSL}_2\,\mathbb{C}$-invariance of the rational normal curve.

The equivalence problem for binary forms

Fundamental covariants

Identify $q \in \mathcal{P}_N$ (polynomial) and $Q(x, y) = y^N q(x/y)$ (binary form).

- $H = (Q, Q)^{(2)}$ (Hessian), $T = (Q, H)^{(1)}$ and $U = (Q, T)^{(1)}$
- $J^2 = T^2/H^3$ and $K = U/H^2$
- Here $(Q, R)^{(r)}(x, y) = \sum_{i=0}^{r} (-1)^r \dfrac{\partial^r Q}{\partial x^{r-i} \partial y^i} \dfrac{\partial^r R}{\partial x^i \partial y^{r-i}}$ is the rth transvectant.

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Theorem (Kogan,Olver)

Two binary forms $Q, \hat{Q}$ are $\mathrm{SL}_2 \mathbb{C}$ equivalent if either $H = \hat{H} = 0$ or there exists a fractional linear transformation $z = (a\hat{z} + b)/(c\hat{z} + d)$ such that $J^2(z) = \hat{J}^2(\hat{z})$ and $K(z) = \hat{K}(\hat{z})$.
Furthermore, the group of symmetries $\mathrm{Aut}(Q)$ is 2-dimensional (translations and scalings) if $H = 0$, and 1-dimensional (scalings only) if $H \neq 0$ but J^2 is a constant. It is finite in all other cases.

The finite symmetry reduction

Proposition

Let $U \in \mathcal{G}_k\mathcal{P}_n$ *be a polynomial subspace and* $Q = \mathrm{W}(U) \in \mathcal{G}_1\mathcal{P}_{k\ell}$ *the Wronskian covariant. Then,* $\mathrm{Aut}\, U \subset \mathrm{Aut}\, Q$*.*

Theorem

- *Let* $U, \hat{U} \in \mathcal{G}_k\mathcal{P}_n$ *be polynomial subspaces,* $Q, \hat{Q}$ *the corresponding Wronksians and* $H, J, K, \hat{H}, \hat{J}, \hat{K}$ *the fundamental covariants of the Wronskians.*
- *Suppose that* $H, \hat{H} \neq 0$ *and that* $J, \hat{J}$ *are not constants.*
- *Then* $U \sim \hat{U}$ *if and only if* $J^2(z) = \hat{J}^2(\hat{z})$ *and* $K(z) = \hat{K}(\hat{z})$ *for some fractional linear transformation* $\hat{z} = g_1(z)$ *and if* $\hat{U} = g_2 \cdot g_1 \cdot U$ *for some* $g_2 \in \mathrm{Aut}\, U$ *(finite).*

Infinite symmetries

Definition

Say that $U \in \mathcal{G}_k\mathcal{P}_n$ is a monomial subspace if $U = \text{span}\{z^{\nu_1}, \dots, z^{\nu_k}\}$, where $0 \leq \nu_1 < \cdots < \nu_k \leq n$.

Theorem

The following are equivalent:

- *U is projectively equivalent to a monomial subspace.*
- *$Q = \mathrm{W}(U)$ has one or two distinct roots;*
- *$H[Q] = 0$ or $J^2[Q]$ is constant;*
- *$\mathrm{Aut}(Q)$ is infinite.*

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Theorem

The following are equivalent:

- $U \sim \mathcal{P}_{k-1}$;
- $Q = \mathrm{W}(U)$ *has one distinct root;*
- $H[Q] = 0$;
- $\operatorname{Aut}(Q)$ *is infinite with* $\dim = 2$.

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- Question: does there exist a polynomial subspace U such that Aut U is strictly smaller than Aut $W(U)$?
- The notion of apolarity is also relevant
  Application: classify all subspaces $\mathcal{G}_k\mathcal{P}_n$ that admit a basis of nth powers.

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- Thank You!