On the preservation of invariants of arc-length type by geometric Hamiltonian curve flows

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Raleigh, April, 2009

Two simple examples

Assume $u : J \subset \mathbb{R}^2 \to \mathbb{RP}^1$ is a solution of the Schwarzian KdV evolution

$$u_t = u_x S(u) = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}$$

where $S(u) = \frac{u_{xxx}}{u_x} - \frac{3}{2}\left(\frac{u_{xx}}{u_x}\right)^2$ is the Schwarzian derivative of u.

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The Schwarzian KdV flow does not need to preserve any differential invariant since $S(u)_t \neq 0$ in general.

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(Hasimoto, 72) We say the Vortex Filament flow is a 3-dimensional Euclidean geometric realization of the NLS equation. The Vortex Filament flow does preserve arc-length, and hence the differential invariant $k = u_x \cdot u_x$. That is, $k_t = 0$.

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These systems are biHamiltonian, i.e., they are Hamiltonian with respect to two compatible Poisson structures. BiHamiltonian systems posses a family of preserved densities that can be calculated using a recursion operator obtained from the Poisson structures. We say the family integrates the systems.

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The bi-Hamiltonian structures used in the integration of both KdV and NLS can be obtained by reduction as follows:

The reduction process

There are two main Poisson brackets in $\mathcal{L}\mathfrak{g}^* = C^\infty(S^1, \mathfrak{g}^*)$ given by

$$\{\mathcal{H}, \mathcal{G}\}_1 = \int_{S^1} \mathrm{tr}\left(\left(\left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)_x + \left[L, \frac{\delta\mathcal{H}}{\delta L}(L)\right]\right)\frac{\delta\mathcal{G}}{\delta L}(L)\right) dx$$

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for any $L_0 \in \mathfrak{g}^*$.
They are compatible and hence they are a biHamiltonian pair.

Theorem

(MB 06) If G is semisimple, the subset of $\mathcal{L}\mathfrak{g}$ generated by Maurer-Cartan matrices for curves in G/H is a quotient of the form $U/\mathcal{L}H$, where $U \subset \mathcal{L}\mathfrak{g}$ is open.

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If $G \subset \mathrm{GL}(n, \mathbb{R})$, the subset of $\mathcal{L}\mathfrak{g}$ generated by the semisimple component of the Maurer-Cartan matrices for curves in $G \ltimes \mathbb{R}^n/G$ can also be identified with $U/\mathcal{L}H$, for some open U, assuming that the nonsemisimple part of the Maurer-Cartan matrices is constant.

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In all known cases associated to an integrable system, the Poisson bracket (2) is also reducible for some good choice of L_0.
That is, the reduction of (2) seems to indicate the existence of an integrable system.

And their geometric realizations...

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(MB 04-06) Let

$$\mathbf{k}_t = F(\mathbf{k}, \mathbf{k}_x, \mathbf{k}_{xx}, \dots) = \mathcal{P}\frac{\delta h}{\delta \mathbf{k}} \tag{3}$$

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In the case $G \ltimes \mathbb{R}^n/G$, some simple condition on $\frac{\delta h}{\delta \mathbf{k}}$ needs to be satisfied to obtain the same result.

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At which level can we see the need to preserve the parameter?

In all known examples, one can already see it at the Poisson structure level, i.e., it is imposed by the background geometry of the flow.

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If $M = Sp(n) \ltimes \mathbb{R}^{2n} / Sp(n)$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, then $k = u_x^T J u_{xx}$ is an invariant of arc-length type.

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If $M = O(n+1,1)/H$ is the Möbius sphere and $p_{ij} = \frac{u^{(i)} \cdot u^{(j)}}{u_x \cdot u_x}$, then

$$k = p_{33} - 6p_{12}p_{23} - p_{13}^2 + 6p_{12}^2 p_{13} + 9p_{12}^2 p_{22} - 5p_{12}^4$$

is an invariant of arc-length type (the parameter generated by its fourth root is the conformal arc-length – Fialkow 42).

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For example, if $\mathbb{RP}^1 = \mathrm{PSL}(2,\mathbb{R})/H$ where H is given by upper triangular matrices, then there exists only one generating differential invariant of the action, the Schwarzian derivative

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and

$$\phi^* S(u) = \left((\phi_x)^2 S(u) + S(\phi)\right) \circ \phi^{-1}.$$

The affine case

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(MB 08) Let $M = G \ltimes \mathbb{R}^n / G$ *be a classical affine geometry (*G *is a classical group) and let* $G \neq \mathrm{GL}(n, \mathbb{R})$*.*

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Corollary

If a system is Hamiltonian with respect to the reduced Poisson bracket, and the evolution

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The case G/H, G semisimple

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But the reduction of (2) is.

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It has a second geometric realization in the centro-affine plane ($\mathrm{SL}(2,\mathbb{R})$ acting linearly on $\mathbb{R}^2$) as flows of star-shaped curves

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Theorem

(A. Calini, T. Ivey, MB 09) The projectivization map induces a local bi-Poisson map between the space of Maurer-Cartan matrices for unparametrized centro-affine curves (parametrized by centro-affine arc-length), and the space of Maurer-Cartan matrices for parametrized projective curves.