Darboux integrability for 2nd-order hyperbolic equations

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Laplace’s method and his invariants

Method of Lapalce

**Linear equation**

\[ u_{xy} + a(x, y) u_x + b(x, y) u_y + c(x, y) u = 0. \]

\[ \frac{\partial}{\partial x}(u_y + au) + b(u_y + au) - h_0 u = 0 \]

\[ h_0 = \frac{\partial a}{\partial x} + ab - c \]

\[ u_1 = u_y + au \]

\[ u_1 x + bu_1 - h_0 u = 0 \]
If $h_0 = 0$, then

$$u_1 = e^{-\int b(x,y)dx},$$

and so

$$u = e^{-\int a(x,y)dy} \int e^{\int a(x,y)dy - \int b(x,y)dx} dy.$$  

If $h_0 \neq 0$, then

$$u = \frac{1}{h_0}(u_1 x + bu_1).$$

Substituting we obtain

$$u_1 = \frac{\partial}{\partial y} \left( \frac{1}{h_0} (u_1 x + bu_1) \right) + a(u_1 x + bu_1),$$
that is,

\[ u_{1xy} + a_1(x, y) u_{1x} + b_1(x, y) u_{1y} + c_1(x, y) u_1 = 0, \]

for some functions \( a_1, b_1 \) and \( c_1 \). We then write the last equation in the form

\[ \frac{\partial}{\partial x} (u_1 y + a_1 u_1) + b_1 (u_1 y + a_1 u_1) - h_1 u_1 = 0, \]

where

\[ h_1 = \frac{\partial a_1}{\partial x} + a_1 b_1 - c_1. \]
If $h_1 = 0$, then general solution $u_1$ for

$$\frac{\partial}{\partial x}(u_1y + a_1u_1) + b_1(u_1y + a_1u_1) - h_1u_1 = 0,$$

can be obtained by quadratures and, using

$$u = \frac{1}{h_0}(u_{1x} + bu_1).$$

we arrive the general solution the original equation without any further integration. If $h_1 \neq 0$, we can repeat the above process to define $h_2$, and so on. Thus we obtain a sequence of functions $h_0, h_1, h_2, \ldots$. If this sequence terminates, that is, if $h_p = 0$ is the last term of the sequence, then the general solution of the original equation can be expressed in terms of quadratures.
Likewise, interchanging $x$ and $y$ we obtain another sequence $k_0, k_1, k_2, \ldots$. If this sequence terminates, that is, if $k_q = 0$ is the last term of the sequence, then the equation

$$u_{xy} + a(x, y) u_x + b(x, y) u_y + c(x, y) u = 0.$$ 

can be integrated by quadratures.
If one of the sequences $h_0, h_1, h_2, \ldots$, or $k_0, k_1, k_2, \ldots$, terminates, we say the equation

$$u_{xy} + a(x, y) u_x + b(x, y) u_y + c(x, y) u = 0.$$ 

is \textit{integrable by the method of Laplace}. If both sequences terminate, it is possible to write a general solution of the equation in terms of two arbitrary functions of one variable and its derivatives, without quadratures (Goursat, circa 1890); in this case we say the equation is \textit{Darboux integrable}. 
Loosely speaking, a system of partial differential equations with equation manifold $\mathcal{R}$ is called \textit{Darboux integrable} if there are \textit{sufficiently many} conservation laws, i.e. sufficiently many functionally independent functions $I$ such that $dl = 0$ on $\mathcal{R}^\infty$. 
Consider the trivial bundle $E = \{ \pi : E \to M \}$, with local coordinates $\pi : (x, y, u) \to (x, y)$. Let $\mathcal{R} \subseteq J^2(E)$ be the equation manifold determined by the second-order hyperbolic equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$ 

Let $\mathcal{R}^\infty$ denotes the infinitely prolonged equation manifold $\mathcal{R}$. With $\mathcal{R}$ we associate the characteristic equation

$$\frac{\partial F}{\partial u_{xx}} \lambda^2 - \frac{\partial F}{\partial u_{xy}} \lambda \mu + \frac{\partial F}{\partial u_{yy}} \mu^2 = 0$$

with positive discriminant

$$\Delta = \left( \frac{\partial F}{\partial u_{xy}} \right)^2 - 4 \frac{\partial F}{\partial u_{xx}} \frac{\partial F}{\partial u_{yy}} > 0.$$
At every point of the equation manifold $\mathcal{R}$. There are two non-proportional real roots $(\mu, \lambda) = (m_x, m_y)$ and $(\mu, \lambda) = (n_x, n_y)$ of equation. The total vector fields

$$X = m_x D_x + m_y D_y, \quad \text{and} \quad Y = n_x D_x + n_y D_y.$$ 

are called the *characteristic vector fields* and they form a basis for the space of total vector fields on $\mathcal{R}^\infty$. 

Characteristic vector fields
Horizontal and vertical differentials

Let $\sigma$ and $\tau$ denote the horizontal forms dual to $X$ and $Y$, i.e.

$$
\sigma(X) = 1, \quad \sigma(Y) = 0, \quad \tau(X) = 0, \quad \tau(Y) = 1.
$$

The forms $\{\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \ldots\}$ define a coframe on $R^\infty$ called the Laplace-adapted coframe.

The exterior derivative $d$ on $R^\infty$ splits into two components

$$
d = d_H + d_V,
$$

where

$$
d_H \omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega).
$$
Let $\mathcal{R}$ be a second-order scalar hyperbolic partial differential equation in the plane with characteristic vector fields $X$ and $Y$. $\mathcal{R}$ is called *Darboux integrable* if for sufficiently large $k$ the characteristic Pfaffian systems

$$C_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k)$$

and

$$C_k(Y) = \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k).$$

each contains a completely integrable subsystem of dimension 2. This implies that there are functions $I, \tilde{I}, J, \tilde{J}$ on $\mathcal{R}^\infty$, such that

$$X(I) = X(\tilde{I}) = 0 \quad \text{and} \quad Y(J) = Y(\tilde{J}) = 0,$$

$$dI \wedge d\tilde{I} \neq 0 \quad \text{and} \quad dJ \wedge d\tilde{J} \neq 0.$$
Laplace adapted coframe on $\mathcal{R}^\infty$

### Universal linearization

\[
\frac{\partial F}{\partial u_{xx}} \theta_{xx} + \frac{\partial F}{\partial u_{xy}} \theta_{xy} + \frac{\partial F}{\partial u_{yy}} \theta_{yy} + \frac{\partial F}{\partial u_x} \theta_x + \frac{\partial F}{\partial u_y} \theta_y + \frac{\partial F}{\partial u} \theta = 0.
\]

### We write

\[
XY(\Theta) + AX(\Theta) + BY(\Theta) + C \Theta = 0,
\]

where $\Theta = \rho \theta$, $\rho = \rho(x, y, u, u_x, u_y)$. 
Laplace adapted coframe on $\mathcal{R}^\infty$

The Laplace-adapted coframe on $\mathcal{R}^\infty$ is constructed by successive applications of the generalized Laplace transform. We define a contact 1-form $\eta_1$ by

$$\eta_1 = Y(\Theta) + A \Theta,$$

and we set

$$H_0 = X(A) + AB - C.$$

We write

$$XY(\Theta) + AX(\Theta) + BY(\Theta) + C \Theta = 0,$$

as

$$X(\eta_1) + B \eta_1 - H_0 \Theta = 0.$$
Laplace adapted coframe on $\mathcal{R}^\infty$

If $H_0 \neq 0$, we have

$$\Theta = \frac{1}{H_0}(X(\eta_1) + B \eta_1)$$

Substituting this into

$$\eta_1 = Y(\Theta) + A \Theta,$$

we arrive at the equation

$$XY(\eta_1) + A_1 X(\eta_1) + B_1 Y(\eta_1) + C_1 \eta_1 = 0,$$

for some functions $A_1$, $B_1$, and $C_1$. We observe that this equation is formally the same as the original universal linearization and therefore we may repeat the process.
Define

\[ \eta_2 = Y(\eta_1) + A_1 \eta_1, \]

and set

\[ H_1 = X(A_1) + A_1 B_1 - C_1. \]

We write

\[ XY(\eta_1) + A_1 X(\eta_1) + B_1 Y(\eta_1) + C_1 \eta_1 = 0, \]

as

\[ X(\eta_2) + B_1 \eta_2 - H_1 \eta_1 = 0. \]

We continue to define \( \eta_3, \eta_4, \ldots \). This process continues until \( H_p = 0 \), for some \( p \) in which case we define inductively

\[ \eta_{p+i+1} = Y(\eta_{p+i}) \quad \text{for all} \quad i \geq 1. \]
Similarly, we construct the other half. We rewrite the universal linearization

$$\frac{\partial F}{\partial u_{xx}} \theta_{xx} + \frac{\partial F}{\partial u_{xy}} \theta_{xy} + \frac{\partial F}{\partial u_{yy}} \theta_{yy} + \frac{\partial F}{\partial u_x} \theta_x + \frac{\partial F}{\partial u_y} \theta_y + \frac{\partial F}{\partial u} \theta = 0.$$ 

as

$$YX(\Theta) + DX(\Theta) + EY(\Theta) + G\Theta.$$

We define a contact 1-form $\xi_1$ by

$$\xi_1 = X(\Theta) + E\Theta$$

and we set

$$K_0 = Y(E) + ED - G.$$
**Suppose that** $H_p = 0$ **and let** $[X, Y] = PX + QY$. **Then**

$$d_H\sigma = -P\sigma \wedge \tau, \quad d_H\tau = -Q\sigma \wedge \tau,$$

$$d_H(\Theta) = \sigma \wedge (\xi_1 - E\Theta) + \tau \wedge (\eta_1 - A\Theta),$$

$$d_H\eta_1 = \sigma \wedge (-B_1\eta_1 + H_0\Theta) + \tau \wedge (\eta_2 - A_1\eta_1),$$

$$d_H\eta_i = \sigma \wedge (-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i\eta_i), \quad 2 \leq i \leq p,$$

$$d_H\eta_{p+1} = \sigma \wedge (-B_p\eta_{p+1}) + \tau \wedge \eta_{p+2},$$

$$d_H\eta_{p+i} = \sigma \wedge \nu_{p+i} + \tau \wedge \eta_{p+i+1} \quad i \geq 2.$$
Suppose that $K_q = 0$. Then

\[ d_H \xi_1 = \tau \wedge (-D \xi_1 + K_0 \Theta) + \sigma \wedge (\xi_2 - E_1 \xi_1), \]

\[ d_H \xi_i = \tau \wedge (-D_{j-1} \xi_j + K_{j-1} \xi_{j-1}) + \sigma \wedge (\xi_{j+1} - E_j \xi_j), \]

for $2 \leq j \leq q$.

\[ d_H \xi_{q+1} = \tau \wedge (-D_q \xi_{q+1}) + \sigma \wedge \xi_{q+2}, \]

\[ d_H \xi_{q+j} = \tau \wedge \mu_{q+j} + \sigma \wedge \xi_{q+j+1}, \]

for $j \geq 2$. In the above equation, $\mu_{q+j}$ is a contact one form such that

\[ \mu_{q+j} \equiv [-(j - 1)P - D_q] \xi_{q+j} \mod \{ \xi_{q+1}, \ldots, \xi_{q+j-1} \}. \]
The $d_V$ structure equations for the horizontal forms $\sigma$ and $\tau$ are

\[ d_V \sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha \quad \text{and} \quad d_V \tau = \sigma \wedge \beta + \tau \wedge \mu_2, \]

where $\alpha, \beta, \mu_1, \mu_2$ are contact 1-forms. The adapted order of $\alpha$ and $\beta$ is $\leq 2$. Moreover, the following relations hold:

\[ d_V P = X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha, \]

\[ d_V Q = X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta, \]

and

\[ d_V \beta = \beta \wedge (\mu_2 - \mu_1), \quad d_V \mu_2 = \alpha \wedge \beta = -d_V \mu_1, \]

\[ d_V \alpha = \alpha \wedge (\mu_1 - \mu_2). \]
\( d_V \) structure equations

\[
\begin{align*}
    d_V \Theta & \equiv 0 \pmod{\{ \Theta \}}; \\
    d_V \eta_i & \equiv 0 \pmod{\{ \xi_1, \Theta, \eta_1, \ldots, \eta_i \}} \quad i \geq 1; \\
    d_V \xi_i & \equiv 0 \pmod{\{ \eta_1, \Theta, \xi_1, \ldots, \xi_i \}} \quad i \geq 1.
\end{align*}
\]

**Proof.** Use induction and the recursive formulas.
Assumption

For the rest of this talk assume \( p \geq 2 \):

\[
H_0 \neq 0, \ H_1 \neq 0, \ldots, \ H_{p-1} \neq 0,
\]

\[
H_p = 0.
\]
**First invariants**

**Theorem 1.**

Let

\[ dl \in C_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k) \]

Then there is a function \( J \) such that \( dJ \in C_k(X) \) and \( mdJ = \tau - \Sigma \) for some function \( m \) and some contact form \( \Sigma \).

**Proof.** Use \( d_H \)-structure equations.

Hence,

\[ d(\tau - \Sigma) \equiv 0 \mod \{\tau - \Sigma\}. \]
First invariants

**Theorem 2.**

For some contact form $\Sigma$

$$d(\tau - \Sigma) \equiv 0 \mod \{\tau - \Sigma\}.$$  

iff

$$X(\Sigma) + Q\Sigma - \beta = 0, \text{ and } d_V(\Sigma) - \sigma \wedge (\mu_2 - Y(\Sigma)) = 0$$

*Proof.* Use $d_H$-structure equations.
Lemma 1.

If $\Sigma$ is a contact form such that $X(\Sigma) + Q\Sigma - \beta = 0$, then

$$\omega = d_V(\Sigma) - \sigma \wedge (\mu_2 - Y(\Sigma))$$

is a relative $X$-invariant form, namely $X(\omega) = -Q\omega$.

Proof. Follows by computation using

$$d_V\sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha$$
$$d_V\tau = \sigma \wedge \beta + \tau \wedge \mu_2,$$

and

$$d_V[X(\omega)] - X(d_V\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega),$$
$$d_V[Y(\omega)] - Y(d_V\omega) = \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega).$$
First invariants

Lemma 2.

Let $l$ be a nonnegative integer, let $\omega$ be a contact form. Suppose $H_p = 0$ and

$$X(\omega) \equiv \lambda \omega \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}.$$ 

Then $\omega$ decomposes uniquely into a sum

$$\omega = \omega_1 + \omega_2,$$

where $\omega_1 \equiv 0 \mod \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}$ and

$$\omega_2 \in \Omega^* (\eta_{p+l+1}, \eta_{p+l+2}, \ldots).$$

Proof. Use $d_H$-structure equations.
Theorem 3.

Let $\Sigma$ be a contact form on $\mathcal{R}^\infty$. Then

$$d(\tau - \Sigma) = 0$$

iff

$$X(\Sigma) + Q\Sigma - \beta = 0.$$ 

Proof. Follows from Lemma 1, Lemma 2, Theorem 2 and $d_V$ and $d_H$-structure equations.
First invariants

Write

\[ \beta = c_0 \Theta + c_1 \eta_1 + c_2 \eta_2 + b_1 \xi_1 + b_2 \xi_2. \]

By using the \( d_H \) structure equations it is not difficult to prove that \( \Upsilon \) is given explicitly by

\[ \Upsilon = b_2 \xi_1 + F_0 \Theta + \sum_{i=1}^{p} F_i \eta_i, \]

where

\[ F_0 = -X(G_1) + (E_1 - Q)G_1 + b_1, \quad F_{i+1} = -\frac{1}{H_i}(X(F_i) - B_i F_i - c_i), \]

and where \( c_i = 0 \) for \( i \geq 3 \).
Lemma 3.

If $H_p = 0$, then there is a unique form

$$\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_p),$$

such that

$$d_V \eta_{p+1} \equiv \eta_{p+2} \wedge \Upsilon \mod \eta_{p+1}$$

and

$$X(\Upsilon) + Q\Upsilon - \beta = 0$$

Proof. Follows from

$$d_V[X(\omega)] - X(d_V \omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega),$$

and Lemma 2.
A second-order hyperbolic scalar equation in the plane,

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \]

is Darboux integrable if and only if for some integers \( p \geq 0 \) and \( q \geq 0 \), the generalized Laplace invariants \( H_p \) and \( K_q \) vanish.

Proof. Follows from Theorem 2 and 3 and the \( d_H \)-structure equations.