Darboux integrability for 2nd-order hyperbolic equations

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Method of Lapalce

Linear equation

$$u_{xy} + a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = 0.$$

$$\frac{\partial}{\partial x}(u_y + au) + b(u_y + au) - h_0 u = 0$$

$$h_0 = \frac{\partial a}{\partial x} + ab - c$$

$$u_1 = u_y + au$$

$$u_{1\,x} + bu_1 - h_0 u = 0$$

Method of Lapalce

If $h_0 = 0$, then

$$u_1 = e^{-\int b(x,y)\mathrm{dx}},$$

and so

$$u = e^{-\int a(x,y)\mathrm{dy}} \int e^{\int a(x,y)\mathrm{dy} - \int b(x,y)\mathrm{dx}}\,\mathrm{dy}.$$

If $h_0 \neq 0$, then

$$u = \frac{1}{h_0}(u_{1x} + bu_1).$$

Substituting we obtain

$$u_1 = \frac{\partial}{\partial y}\left(\frac{1}{h_0}(u_{1x} + bu_1)\right) + a(u_{1x} + bu_1),$$

Method of Lapalce

that is,

$$u_{1\,xy} + a_1(x,y)\,u_{1\,x} + b_1(x,y)\,u_{1\,y} + c_1(x,y)\,u_1 = 0,$$

for some functions a_1, b_1 and c_1. We then write the last equation in the form

$$\frac{\partial}{\partial x}(u_{1\,y} + a_1 u_1) + b_1(u_{1\,y} + a_1 u_1) - h_1 u_1 = 0,$$

where

$$h_1 = \frac{\partial a_1}{\partial x} + a_1 b_1 - c_1 .$$

Method of Lapalce

If $h_1 = 0$, then general solution u_1 for

$$\frac{\partial}{\partial x}(u_{1y} + a_1 u_1) + b_1(u_{1y} + a_1 u_1) - h_1 u_1 = 0,$$

can be obtained by quadratures and, using

$$u = \frac{1}{h_0}(u_{1x} + b u_1).$$

we arrive the general solution the original equation without any further integration. If $h_1 \neq 0$, we can repeat the above process to define h_2, and so on. Thus we obtain a sequence of functions $h_0, h_1, h_2, \ldots$. If this sequence terminates, that is, if $h_p = 0$ is the last term of the sequence, then the general solution of the original equation can be expressed in terms of quadratures.

Method of Lapalce

Likewise, interchanging x and y we obtain another sequence $k_0, k_1, k_2, \ldots$. If this sequence terminates, that is, if $k_q = 0$ is the last term of the sequence, then the equation

$$u_{xy} + a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = 0.$$

can be integrated by quadratures.

Method of Lapalce

If one of the sequences $h_0, h_1, h_2, \ldots,$ or $k_0, k_1, k_2, \ldots,$ terminates, we say the equation

$$u_{xy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0.$$

is *integrable by the method of Laplace.* If both sequences terminate, it is possible to write a general solution of the equation in terms of two arbitrary functions of one variable and its derivatives, without quadratures (Goursat, circa 1890); in this case we say the equation is *Darboux integrable.*

Darboux integrability in general

Loosely speaking, a system of partial differential equations with equation manifold $\mathcal{R}$ is called *Darboux integrable* if there are *sufficiently many* conservation laws, i.e. sufficiently many functionally independent functions I such that $dI = 0$ on $\mathcal{R}^\infty$.

Equation manifold

Consider the trivial bundle $E = \{\pi : E \to M\}$, with local coordinates $\pi : (x, y, u) \to (x, y)$. Let $\mathcal{R} \subseteq J^2(E)$ be the equation manifold determined by the second-order hyperbolic equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Let $\mathcal{R}^\infty$ denotes the infinitely prolonged equation manifold $\mathcal{R}$. With $\mathcal{R}$ we associate the *characteristic equation*

$$\frac{\partial F}{\partial u_{xx}}\lambda^2 - \frac{\partial F}{\partial u_{xy}}\lambda\mu + \frac{\partial F}{\partial u_{yy}}\mu^2 = 0$$

with positive *discriminant*

$$\Delta = \left(\frac{\partial F}{\partial u_{xy}}\right)^2 - 4\frac{\partial F}{\partial u_{xx}}\frac{\partial F}{\partial u_{yy}} > 0.$$

Characteristic vector fields

At every point of the equation manifold $\mathcal{R}$. There are two non-proportional real roots $(\mu, \lambda) = (m_x, m_y)$ and $(\mu, \lambda) = (n_x, n_y)$ of equation. The total vector fields

$$X = m_x D_x + m_y D_y, \quad \text{and} \quad Y = n_x D_x + n_y D_y.$$

are called the *characteristic vector fields* and they form a basis for the space of total vector fields on $\mathcal{R}^\infty$.

Horizontal and vertical differentials

Let σ and τ denote the horizontal forms dual to X and Y, i. e.

$$\sigma(X) = 1,\ \sigma(Y) = 0,\ \tau(X) = 0,\ \tau(Y) = 1.$$

The forms $\{\, \sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \dots \}$ define a coframe on $\mathcal{R}^\infty$ called the *Laplace-adapted coframe.*

The exterior derivative d on $\mathcal{R}^\infty$ splits into two components

$$d = d_H + d_V,$$

where

$$d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega).$$

Darboux integrability

Let $\mathcal{R}$ be a second-order scalar hyperbolic partial differential equation in the plane with characteristic vector fields X and Y. $\mathcal{R}$ is called *Darboux integrable* if for sufficiently large k the characteristic Pfaffian systems

$$\mathcal{C}_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k)$$

and

$$\mathcal{C}_k(Y) = \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k).$$

each contains a completely integrable subsystem of dimension 2. This implies that there are functions I, $\tilde{I}$, J, $\tilde{J}$ on $\mathcal{R}^\infty$, such that

$$X(I) = X(\tilde{I}) = 0 \qquad \text{and} \qquad Y(J) = Y(\tilde{J}) = 0,$$

$$dI \wedge d\tilde{I} \neq 0 \qquad \text{and} \qquad dJ \wedge d\tilde{J} \neq 0.$$

Laplace adapted coframe on $\mathcal{R}^\infty$

Universal linearization

$$\frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \frac{\partial F}{\partial u_{xy}}\,\theta_{xy} + \frac{\partial F}{\partial u_{yy}}\,\theta_{yy} + \frac{\partial F}{\partial u_{x}}\,\theta_{x} + \frac{\partial F}{\partial u_{y}}\,\theta_{y} + \frac{\partial F}{\partial u}\,\theta = 0.$$

We write

$$XY(\Theta) + A\,X(\Theta) + B\,Y(\Theta) + C\,\Theta = 0,$$

where $\Theta = \rho\theta$, $\rho = \rho(x, y, u, u_x, u_y)$.

Laplace adapted coframe on $\mathcal{R}^\infty$

The Laplace-adapted coframe on $\mathcal{R}^\infty$ is constructed by successive applications of the generalized Laplace transform. We define a contact 1-form η_1 by

$$\eta_1 = Y(\Theta) + A\,\Theta,$$

and we set

$$H_0 = X(A) + AB - C.$$

We write

$$XY(\Theta) + A\,X(\Theta) + B\,Y(\Theta) + C\,\Theta = 0,$$

as

$$X(\eta_1) + B\,\eta_1 - H_0\,\Theta = 0.$$

Laplace adapted coframe on $\mathcal{R}^\infty$

If $H_0 \neq 0$, we have

$$\Theta = \frac{1}{H_0}(X(\eta_1) + B\,\eta_1)$$

Substituting this into

$$\eta_1 = Y(\Theta) + A\,\Theta,$$

we arrive at the equation

$$XY(\eta_1) + A_1 X(\eta_1) + B_1 Y(\eta_1) + C_1 \eta_1 = 0,$$

for some functions A_1, B_1, and C_1. We observe that this equation is formally the same as the original universal linearization and therefore we may repeat the process.

Laplace adapted coframe on $\mathcal{R}^\infty$

Define

$$\eta_2 = Y(\eta_1) + A_1\,\eta_1,$$

and set

$$H_1 = X(A_1) + A_1 B_1 - C_1.$$

We write

$$XY(\eta_1) + A_1 X(\eta_1) + B_1\,Y(\eta_1) + C_1\eta_1 = 0,$$

as

$$X(\eta_2) + B_1\,\eta_2 - H_1\,\eta_1 = 0.$$

We continue to define $\eta_3, \eta_4, \ldots$. This process continues until $H_p = 0$, for some p in which case we define inductively

$$\eta_{p+i+1} = Y(\eta_{p+i}) \qquad \text{for all } i \geq 1.$$

Laplace adapted coframe on $\mathcal{R}^\infty$

Similarly, we construct the other half. We rewrite the universal linearization

$$\frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \frac{\partial F}{\partial u_{xy}}\,\theta_{xy} + \frac{\partial F}{\partial u_{yy}}\,\theta_{yy} + \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_y}\,\theta_y + \frac{\partial F}{\partial u}\,\theta = 0.$$

as

$$YX(\Theta) + D\,X(\Theta) + E\,Y(\Theta) + G\,\Theta.$$

We define a contact 1-form ξ_1 by

$$\xi_1 = X(\Theta) + E\,\Theta$$

and we set

$$K_0 = Y(E) + ED - G.$$

$$\ldots.$$

d_H structure equations

Suppose that $H_p = 0$ and let $[X, Y] = PX + QY$. Then

$$d_H\sigma = -P\,\sigma \wedge \tau, \qquad d_H\tau = -Q\,\sigma \wedge \tau,$$

$$d_H(\Theta) = \sigma \wedge (\xi_1 - E\,\Theta) + \tau \wedge (\eta_1 - A\,\Theta),$$

$$d_H\eta_1 = \sigma \wedge (-B\,\eta_1 + H_0\Theta) + \tau \wedge (\eta_2 - A_1\eta_1),$$

$$d_H\eta_i = \sigma \wedge (-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i\eta_i),\ 2 \leq i \leq p,$$

$$d_H\eta_{p+1} = \sigma \wedge (-B_p\eta_{p+1}) + \tau \wedge \eta_{p+2},$$

$$d_H\eta_{p+i} = \sigma \wedge \nu_{p+i} + \tau \wedge \eta_{p+i+1} \quad i \geq 2.$$

In the last equation, ν_{p+i} is a contact one form such that

$$\nu_{p+i} \equiv [(i-1)Q - B_p]\,\eta_{p+i} \qquad \text{mod } \{\,\eta_{p+1}, \ldots, \eta_{p+i-1}\,\}.$$

d_H structure equations

Suppose that $K_q = 0$. Then

$$d_H\xi_1 = \tau \wedge (-D\,\xi_1 + K_0\Theta) + \sigma \wedge (\xi_2 - E_1\xi_1),$$

$$d_H\xi_i = \tau \wedge (-D_{j-1}\xi_j + K_{j-1}\xi_{j-1}) + \sigma \wedge (\xi_{j+1} - E_j\xi_j),$$

for $2 \leq j \leq q$.

$$d_H\xi_{q+1} = \tau \wedge (-D_q\xi_{q+1}) + \sigma \wedge \xi_{q+2},$$

$$d_H\xi_{q+j} = \tau \wedge \mu_{q+j} + \sigma \wedge \xi_{q+j+1},$$

for $j \geq 2$. In the above equation, μ_{q+j} is a contact one form such that

$$\mu_{q+j} \equiv [-(j-1)P - D_q]\,\xi_{q+j} \qquad \text{mod } \{\,\xi_{q+1}, \ldots, \xi_{q+j-1}\,\}.$$

d_V structure equations

The d_V structure equations for the horizontal forms σ and τ are

$$d_V\sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha \quad \text{and} \quad d_V\tau = \sigma \wedge \beta + \tau \wedge \mu_2,$$

where α, β, μ_1, μ_2 are contact 1-forms. The adapted order of α and β is ≤ 2. Moreover, the following relations hold:

$$d_V P = X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha,$$

$$d_V Q = X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta,$$

and

$$d_V\beta = \beta \wedge (\mu_2 - \mu_1), \; d_V\mu_2 = \alpha \wedge \beta = -d_V\mu_1,$$

$$d_V\alpha = \alpha \wedge (\mu_1 - \mu_2).$$

d_V structure equations

$$
\begin{aligned}
d_V\Theta &\equiv 0 \quad &&\bmod \{\,\Theta\,\}; \\
d_V\eta_i &\equiv 0 \quad &&\bmod \{\,\xi_1, \Theta, \eta_1, \ldots, \eta_i\,\} \qquad i \geq 1; \\
d_V\xi_i &\equiv 0 \quad &&\bmod \{\,\eta_1, \Theta, \xi_1, \ldots, \xi_i\,\} \qquad i \geq 1.
\end{aligned}
$$

Proof. Use induction and the recursive formulas.

Assumption

For the rest of this talk assume $p \geq 2$:

$$H_0 \neq 0, H_1 \neq 0, \ldots, H_{p-1} \neq 0,$$

$$H_p = 0.$$

First invariants

Theorem 1.

Let

$$dI \in \mathcal{C}_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \dots, \eta_k, \xi_k)$$

Then there is =a function J such that $dJ \in \mathcal{C}_k(X)$ and $mdJ = \tau - \Sigma$ for some function m and some contact form Σ.

Proof. Use d_H-structure equations.

Hence,

$$d(\tau - \Sigma) \equiv 0 \mod \{\tau - \Sigma\}.$$

First invariants

Theorem 2.

For some contact form Σ

$$d(\tau - \Sigma) \equiv 0 \mod \{\tau - \Sigma\}.$$

iff

$$X(\Sigma) + Q\Sigma - \beta = 0, \text{ and } d_V(\Sigma) - \sigma \wedge (\mu_2 - Y(\Sigma)) = 0$$

Proof. Use d_H-structure equations.

First invariants

Lemma 1.

If Σ is a contact form such that $X(\Sigma) + Q\Sigma - \beta = 0$, then

$$\omega = d_V(\Sigma) - \sigma \wedge (\mu_2 - Y(\Sigma))$$

is a relative X-invariant form, namely $X(\omega) = -Q\omega$.
Proof. Follows by computation using

$$d_V\sigma = \sigma \wedge \mu_1 + \tau \wedge \alpha$$

$$d_V\tau = \sigma \wedge \beta + \tau \wedge \mu_2,$$

and

$$d_V[X(\omega)] - X(d_V\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega),$$
$$d_V[Y(\omega)] - Y(d_V\omega) = \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega).$$

First invariants

Lemma 2.

Let l be a nonnegative integer, let ω be a contact form. Suppose $H_p = 0$ and

$$X(\omega) \equiv \lambda\,\omega \qquad \text{mod } \{\,\eta_{p+1}, \eta_{p+2}, \dots, \eta_{p+l}\,\}.$$

Then ω decomposes uniquely into a sum

$$\omega = \omega_1 + \omega_2,$$

where $\omega_1 \equiv 0 \mod \{\,\eta_{p+1}, \eta_{p+2}, \dots, \eta_{p+l}\,\}$ and

$$\omega_2 \in \Omega^*(\eta_{p+l+1}, \eta_{p+l+2}, \dots).$$

Proof. Use d_H-structure equations.

First invariants

Theorem 3.

Let Σ be a contact form on $\mathcal{R}^\infty$. Then

$$d(\tau - \Sigma) = 0$$

iff

$$X(\Sigma) + Q\Sigma - \beta = 0.$$

Proof. Follows from Lemma 1, Lemma 2, Theorem 2 and d_V and d_H-structure equations.

First invariants

Write

$$\beta = c_0\Theta + c_1\eta_1 + c_2\eta_2 + b_1\xi_1 + b_2\xi_2.$$

By using using the d_H structure equations it is not difficult to prove that Υ is given explicitly by

$$\Upsilon = b_2\xi_1 + F_0\Theta + \sum_{i=1}^{p} F_i\eta_i,$$

where

$$F_0 = -X(G_1)+(E_1-Q)G_1+b_1, \quad F_{i+1} = -\frac{1}{H_i}(X(F_i)-B_iF_i-c_i),$$

and where $c_i = 0$ for $i \geq 3$.

First invariants

Lemma 3.

If $H_p = 0$, then there is a unique form

$$\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_p),$$

such that

$$d_V \eta_{p+1} \equiv \eta_{p+2} \wedge \Upsilon \quad \text{mod } \eta_{p+1}$$

and

$$X(\Upsilon) + Q\Upsilon - \beta = 0$$

Proof. Follows from

$$d_V[X(\omega)] - X(d_V\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega),$$

and Lemma 2.

Main theorem

A second-order hyperbolic scalar equation in the plane,

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is Darboux integrable if and only if for some integers $p \geq 0$ and $q \geq 0$, the generalized Laplace invariants H_p and K_q vanish.

Proof. Follows from Theorem 2 and 3 and the d_H-structure equations.