

Maximally symmetric hyperbolic equations of generic type

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Outline of main points

- 1 Definition and examples of hyperbolic 6-6, 6-7, 7-7 eqns.
- 2 (Vranceanu 1937) Hyperbolic 7-7 eqns have at most 9-dim symmetry. \exists 1-parameter family of such max. sym. models.
- 3 (T. 2008) These max. sym. hyp 7-7 models have striking connections to Cartan's G_2 models (3 equivalent realizations). This remains to be fully explored.

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$$

Work in $J^2(\mathbb{R}^2, \mathbb{R}) : (x, y, z, p, q, r, s, t)$ with contact system \mathcal{C}^2 :

$$\theta^1 = dz - pdx - qdy, \quad \theta^2 = dp - rdx - sdy, \quad \theta^3 = dq - sdx - tdy$$

Parametrize $F = 0$: $i_F : \Sigma^7 \rightarrow J^2(\mathbb{R}^2, \mathbb{R})$

Define: $I_F = i_F^*(\mathcal{C}^2)$

Assume: [1] i_F is max. rank, [2] $(F_r, F_s, F_t) \neq 0$, [3] $dF|_{F=0} \neq 0$

Definition

$F = 0$ and $\bar{F} = 0$ are *contact-equiv.* if \exists local diffeo. $\phi : \Sigma \rightarrow \bar{\Sigma}$ such that $\phi^* I_{\bar{F}} = I_F$. A *contact symmetry* is a self-equivalence.

General goal: Find invariants distinguishing equiv. classes.

Elliptic, parabolic, hyperbolic

Gardner (1969): Conformal sym. $C^\infty(\Sigma)$ -bilinear form $\langle \cdot, \cdot \rangle$ on I_F :

$$\langle \varphi, \psi \rangle \text{Vol}_\Sigma := d\varphi \wedge d\psi \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \quad \forall \varphi, \psi \in I_F$$

where $\omega^1, \omega^2, \omega^3$ generate I_F . W.r.t. some basis, $\langle \cdot, \cdot \rangle$ is given by

$$i_F^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & F_t & -\frac{1}{2}F_s \\ 0 & -\frac{1}{2}F_s & F_r \end{pmatrix}.$$

Defining $\Delta = i_F^* (F_r F_t - \frac{1}{4} F_s^2)$, we have (pointwise):

elliptic	parabolic	hyperbolic
$\Delta > 0$	$\Delta = 0$	$\Delta < 0$

This classification is *contact-invariant*.

Hyperbolic equations: 6-6, 6-7, 7-7

Hyperbolic case: \exists pair of rank 2 max. isotropic subsystems

$$M_1, M_2 \subset I_F$$

(Invariant up to $M_1 \leftrightarrow M_2$.) Then $\langle \cdot, \cdot \rangle$ has normal form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma

M_1 and M_2 each admit at most one Cauchy characteristic v.f.

Name	M_1, M_2 admit Cauchy char. v.f.?
6-6 (Monge–Ampère)	both
6-7 (Goursat)	exactly one of them
7-7 (generic)	neither

Examples of hyperbolic equations

Examples (Gardner–Kamran 1993)

- all 6-6: $a(rt - s^2) + br + cs + dt + e = 0$, where a, b, c, d, e are functions on $J^1(\mathbb{R}^2, \mathbb{R})$.
- 6-7: $r = f(s)$ where $f'' \neq 0$.
- 7-7: $s = \frac{1}{2} \sin(r) \cos(t)$, or $3rt^3 + 1 = 0$.

Examples (T. 2008)

- $F(x, y, z, p, q, r, t) = 0$: 6-6 or 7-7
- $G(x, y, z, p, q, r, s) = 0$: 6-6 or 6-7
- $rt = f(x, y, z, p, q, s)$: 6-6 or 7-7

Relative contact-invariants for hyperbolic eqns

Vranceanu (1940): invariants for $r = f(x, y, z, p, q, s, t)$

Juráš (1997): General $F = 0$ [Monge–Ampère invariants]

Theorem (T. 2008)

Suppose that $F = 0$ is a hyperbolic equation with $F_s(\sigma) \geq 0$. Let

$$I_1 = i_F^* \det \begin{pmatrix} F_r & F_s & F_t \\ \lambda_+ & F_t & 0 \\ \left(\frac{F_t}{\lambda_+}\right)_r & \left(\frac{F_t}{\lambda_+}\right)_s & \left(\frac{F_t}{\lambda_+}\right)_t \end{pmatrix}, \quad I_2 = i_F^* \det \begin{pmatrix} 0 & F_r & \lambda_+ \\ F_r & F_s & F_t \\ \left(\frac{F_r}{\lambda_+}\right)_r & \left(\frac{F_r}{\lambda_+}\right)_s & \left(\frac{F_r}{\lambda_+}\right)_t \end{pmatrix},$$

where $\lambda_+ = \frac{1}{2}(F_s + \sqrt{|\Delta|}) > 0$. Then we can classify $F = 0$ at σ :

Type	Contact-invariant classification
6-6	$I_1 = I_2 = 0$
6-7	exactly one of I_1 or I_2 is zero
7-7	$I_1 I_2 \neq 0$

Monge subsystems

Usually defined in terms of singular subsystems, involutions, etc. In this case, given M_1 and M_2 , the following defn is simpler:

Definition (Monge subsystems)

Let $\mathcal{D} = I_F^\perp$, $\mathcal{M}_i = M_i^\perp$. The Monge subsystems are $\mathcal{C}_i = \mathcal{C}_i^\perp$,

$$\mathcal{C}_i := \{X \in \mathcal{D} : [X, \mathcal{D}] \subset \mathcal{M}_i\} \quad (\text{rank } 2).$$

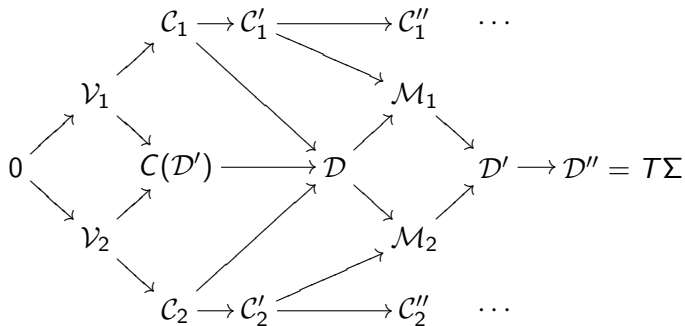
Well-known that can use these to distinguish equivalence classes.

Name	PDE	$\text{rank}(\mathcal{C}_i^{(k)}), k \geq 0$
wave	$z_{xy} = 0$	5, 4, $\bar{3}$
Liouville	$z_{xy} = e^z$	5, 4, 3, $\bar{1}$
Klein–Gordon	$z_{xy} = z$	5, 4, 3, 2, $\bar{1}$

Thus, these 3 equations (in the MA class) are contact-inequivalent.

Hyperbolic 7-7 equations

Vector distribution picture for hyp. 7-7 eqns:



rk : 0 1 2 3 4 5 6 7

Results of the Cartan equivalence method

$\phi \in \text{Contact}(\Sigma, \bar{\Sigma})$ iff $\phi^*\bar{\omega} = g\omega$ for some $g : \Sigma \rightarrow G$.

- 3-dim. structure group: $G = G^0 \rtimes D_8 \subset GL(7, \mathbb{R})$
- Can always reduce the structure group dim. by at least one.
- Either get to an $\{e\}$ -structure on Σ or (since $\mathfrak{g}^{(1)} = 0$) we get to an $\{e\}$ -structure on $\Sigma \times G_\Gamma$, where $\dim(G_\Gamma) \leq 2$.

Theorem (Vranceanu 1937)

The sym. grp of any hyp. 7-7 eqn has dim. ≤ 9 .

FACT: Upper bound is realized, but *not uniquely!*

The maximally symmetric case

Get complete list of contact-inequivalent structures with maximal 9-dim symmetry:

Theorem (Vranceanu 1937, T. 2008)

Normal forms for max. sym. hyp. 7-7 eqns are:

$$\alpha = 0 : \quad 3rt^3 + 1 = 0$$

$$\alpha \in (0, 1) \cup (1, 2] : \quad \frac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} + \frac{(2 - \alpha)^2}{1 - \alpha} = 0$$

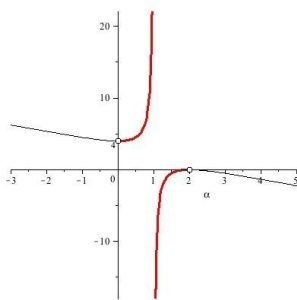
A sanity check!

Note that $J = \frac{(3r-6st+2t^3)^2}{(2s-t^2)^3} = \frac{\sigma^2}{\tau^3}$ is nec. a diff. inv. for the sym. alg. Thus, $F = J + c = 0$ is also invariant $\forall c \in \mathbb{R}$. On $F = 0$,

$$\Delta = F_r F_t - \frac{1}{4} F_s^2 = -\frac{9}{\tau^2} (4J + J^2) = -\frac{9}{\tau^2} c(c - 4)$$

Thus, $F = J + c = 0$ is hyperbolic iff $c < 0$ or $c > 4$.

Does $c(\alpha) = \frac{(2-\alpha)^2}{1-\alpha}$, $\alpha \in (0, 1) \cup (1, 2)$ cover these cases? **YES!**



Symmetries

FACT: For any $\alpha \neq 0$, the symmetry algebra is *independent of* α .

Theorem (T. 2008)

\exists two distinct 9-dim. contact sym. alg. $\mathfrak{h}_0, \mathfrak{h}_1$ for max. sym. hyp. 7-7 eqns. Both are transitive & imprimitive. Abstractly,
$$\mathfrak{h}_\delta \cong \mathfrak{gl}(2, \mathbb{R}) \ltimes \mathfrak{n}_\delta$$

\mathfrak{n}_δ	e_1	e_2	e_3	e_4	e_5
e_1	\cdot	δe_3	e_4	\cdot	\cdot
e_2		\cdot	e_5	\cdot	\cdot
e_3			\cdot	\cdot	\cdot
e_4				\cdot	\cdot
e_5					\cdot

Corollary

A hyp. 7-7 eqn is equiv. to $3rt^3 + 1 = 0$ iff (1) 9-dim sym., (2) $\dim(\mathfrak{n}^{(1)}) = 2$.

NOTE: \mathfrak{h}_1 has a natural 3-step grading preserved by the bracket:

$$\mathfrak{h}_1 = T_0 \oplus T_1 \oplus T_2 \oplus T_3 = \mathfrak{gl}(2, \mathbb{R}) \oplus \{e_1, e_2\} \oplus \{e_3\} \oplus \{e_4, e_5\}$$

satisfying $[T_i, T_j] \subset T_{i+j}$.

Identification of \mathfrak{h}_1 and degeneration to G_2 -invariant eqns

In fact,

\mathfrak{h}_1 is a maximal parabolic subalgebra of \mathfrak{g}_2 . (Delete long root.)

Cartan's 5-var. paper: $\exists!$ max. sym. $(3, 2, 0)$ system in dim. 5 and it has G_2 symmetry. Cartan gives 3 explicit realizations for it:

- 1 parabolic Goursat eqn: $9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0$
- 2 involutive system: $r = \frac{t^3}{3}, s = \frac{t^2}{2}$
- 3 Hilbert–Cartan eqn: $u' = (v'')^2$

For $\alpha \neq 0$, sym. are indep. of α , so eqn degenerations inherit \mathfrak{h}_1 as a sym. subalg. In certain cases, the sym. alg. is strictly larger:

- $\alpha \rightarrow 0$: Get parabolic Goursat model above.
- $\alpha = 2$: This eqn $F = 3r - 6st + 2t^3 = 0$ has $\Delta = -9(2s - t^2)$.
In the parabolic case, get the involutive model above.

Darboux integrability and more on G_2 symmetry

All max. sym. models have $C_i^{(2)}$ rank 2 and *Frobenius*.

Theorem (Vranceanu 1937)

All max. sym. hyp. 7-7 eqns are Darboux integrable.

Write $C_1^{(2)} = \{df_1, df_2\}$, $C_2^{(2)} = \{dg_1, dg_2\}$. Thus, \exists two distinct (local) foliations of Σ^7 by 5-dim. leaves:

- 1 $\mathcal{L}^1 = \{\mathcal{L}_c^1\}_{c \in \mathbb{R}^2}$, $\mathcal{L}_c^1 = \{f_1 = c_1, f_2 = c_2\}$
- 2 $\mathcal{L}^2 = \{\mathcal{L}_c^2\}_{c \in \mathbb{R}^2}$, $\mathcal{L}_c^2 = \{g_1 = c_1, g_2 = c_2\}$

Theorem (T. 2009)

Let $F = 0$ be any max. sym. hyp. 7-7 eqn. Fix any leaf $i : \mathcal{L} \rightarrow \Sigma$ in either \mathcal{L}^1 or \mathcal{L}^2 . Then $i^ I_F$ has derived flag $(3, 2, 0)$, has \mathfrak{g}_2 symmetry, and is equiv. to the Hilbert–Cartan model.*

- 1 Geometric interpretation of α in the max. sym. models. (Disappears when passing to the leaves and leaf space.)
- 2 Is \mathfrak{h}_0 distinguished in any way? (Not a subalg of \mathfrak{g}_2 ; not a parabolic of any s.s. Lie alg; it *is* a subalg of $\mathfrak{sl}(4, \mathbb{R})$.)
- 3 If C_i contains Frobenius subsystems, then we can pull back I_F to the (5 or 6-dim) leaves of the associated foliation. What if not? i.e. “General hyp. 7-7 eqns”. Derived flag is either $(5, 4, 2, 0)$ or $(5, 4, 2, 1, 0)$. Can these both occur? Find max. sym. models. What sym. alg. arise?
- 4 Hyperbolic 6-7 equations? (see Petrescu)