Geometry of control-affine systems

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Motivation: Optimal control theory

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A *control system* may be described in local coordinates by an underdetermined system of ordinary differential equations

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ represents the *state* of the system and $u \in \mathbb{R}^s$ represents the *controls*. More generally, x and u may take values in an n-dimensional manifold $\mathcal{X}$ and and s-dimensional manifold $\mathcal{U}$, respectively.

The system (1) is called *control-linear* if the right-hand side is linear in the control variables u and depends smoothly on the state variables x; i.e., if the system has the form

$$\dot{x} = A(x)u, \tag{2}$$

where $A(x)$ is an $n \times s$ matrix whose entries are smooth functions of x.

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For such a system, *admissible paths* in $\mathcal{X}$ are those for which the tangent vector to the path at each point $x \in \mathcal{X}$ is contained in the subspace $\mathcal{D}_x \subset T_x\mathcal{X}$ determined by the image of the matrix $A(x)$.

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If this matrix has constant (maximal) rank s on $\mathcal{X}$, then $\mathcal{D}$ is a rank s (linear) distribution on $\mathcal{X}$. Thus the admissible paths in the state space are precisely the *horizontal curves* of the distribution $\mathcal{D}$, i.e., curves whose tangent vectors at each point are contained in $\mathcal{D}$.

Given a distribution $(\mathcal{X}, \mathcal{D})$, the iterated brackets of vector fields which are local sections of $\mathcal{D}$ generate a flag of distributions

$$\mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset T\mathcal{X},$$

called the *derived series* of $\mathcal{D}$, defined by

$$\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \qquad i \geq 1.$$

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- The distribution $\mathcal{D}$ is *bracket-generating* if $\mathcal{D}^\infty = T\mathcal{X}$.
- Set $n_i(x) = \dim \mathcal{D}_x^i$. The *growth vector* of $\mathcal{D}$ at x is the integer list $(n_1(x), n_2(x), \ldots, n_r(x))$, where $r(x)$ is the step of $\mathcal{D}$ at x.

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- The distribution $\mathcal{D}$ has *constant type* if $n_i(x)$ is constant on $\mathcal{X}$ for all i; i.e., if the growth vector of $\mathcal{D}$ is constant on $\mathcal{X}$.

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- The distribution $\mathcal{D}$ has *constant type* if $n_i(x)$ is constant on $\mathcal{X}$ for all i; i.e., if the growth vector of $\mathcal{D}$ is constant on $\mathcal{X}$.
- The distribution $\mathcal{D}$ is called *integrable* if $\mathcal{D}^\infty = \mathcal{D}$.

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Theorem 1 *(Frobenius) Let $\mathcal{D}$ be a rank s distribution on an n-dimensional manifold $\mathcal{X}$. If $\mathcal{D}$ is integrable, then in a sufficiently small neighborhood of any point $x \in \mathcal{X}$, there exist local coordinates $(x^1, \ldots, x^n)$ such that*

$$\mathcal{D} = span\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^s}\right), \tag{3}$$

or, equivalently,

$$\mathcal{D}^{\perp} = \{dx^{s+1}, \ldots, dx^n\}. \tag{4}$$

Theorem 2 *(Pfaff) Let* $\mathcal{D}$ *be a rank* n *distribution on an* $(n+1)$*-dimensional manifold* $\mathcal{X}$*, and let* θ *be a nonvanishing 1-form on* $\mathcal{X}$ *such that* $\mathcal{D} = \{\theta\}^\perp$*. Let* k *be the integer defined by the conditions*

$$\theta \wedge (d\theta)^k \neq 0, \qquad \theta \wedge (d\theta)^{k+1} = 0.$$

(k *is called the* Pfaff rank *of* θ*.) In a sufficiently small neighborhood of any point* $x \in \mathcal{X}$ *on which* k *is constant, there exist local coordinates* $(x^0, \ldots, x^n)$ *such that*

$$\theta = \begin{cases} dx^1 + x^2\, dx^3 + \cdots + x^{2k}\, dx^{2k+1} & \text{if } (d\theta)^{k+1} = 0, \\ x^0\, dx^1 + x^2\, dx^3 + \cdots + x^{2k}\, dx^{2k+1} & \text{if } (d\theta)^{k+1} \neq 0. \end{cases} \tag{5}$$

Theorem 3 *(Engel) Let $\mathcal{D}$ be a rank 2 distribution of constant type on a 4-dimensional manifold $\mathcal{X}$, with growth vector $(2, 3, 4)$. Then in a sufficiently small neighborhood of any point $x \in \mathcal{X}$, there exist local coordinates (x^1, x^2, x^3, x^4) such that*

$$\mathcal{D}^\perp = \{dx^2 - x^3\, dx^1,\ dx^3 - x^4\, dx^1\}. \tag{6}$$

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More recently, Bryant has described local invariants of rank 3 distributions on 6-manifolds with growth vector $(3, 6)$, and Doubrov and Zeleneko have given a fairly comprehensive treatment of maximally nonholonomic distributions of ranks 2 and 3 on manifolds of arbitrary dimension.

While the class (2) of control-linear systems contains many interesting examples, a much broader class of interest is that of *control-affine* systems, i.e., systems of the form

$$\dot{x} = a_0(x) + A(x)u \tag{7}$$

where $A(x)$ is a smoothly-varying $n \times s$ matrix and $a_0(x)$ is a smooth vector field on $\mathcal{X}$. (The vector field $a_0(x)$ is known as the *drift* vector field.)

Example 1 The class of control-affine systems contains as a sub-class the *linear* control systems. These are systems of the form

$$\dot{x} = Ax + Bu, \tag{8}$$

where A is a constant $n \times n$ matrix and B is a constant $n \times s$ matrix. Note that despite the terminology, the linear control system (2) is not control-linear unless $A = 0$.

Example 2 Mechanical systems. These systems are typically second-order, e.g.,

$$\ddot{x} = \sum_{i=1}^{s} u^i F_i(x).$$

This system can be rewritten as the following first-order system on the cotangent bundle $T^*\mathcal{X}$, with local coordinates (x, p):

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_i(x) \end{bmatrix} \begin{bmatrix} u^i \end{bmatrix}.$$

Given a control-affine system

$$\dot{x} = a_0(x) + A(x)u,$$

we can canonically associate to it an *affine distribution* $\mathcal{F}$ on the state space $\mathcal{X}$: if we let $a_1(x), \ldots, a_s(x)$ denote the columns of the matrix $A(x)$, then set

$$\mathcal{F}_x = a_0(x) + \operatorname{span}\left(a_1(x), \ldots, a_s(x)\right).$$

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We will say that $\mathcal{F}$ is *properly affine* if none of the affine subspaces $\mathcal{F}_x \subset T_x\mathcal{X}$ are linear subspaces. Associated to an affine distribution $\mathcal{F}$ is the *direction distribution*

$$L_{\mathcal{F}} = \{\xi_1 - \xi_2 \mid \xi_1, \xi_2 \in \mathcal{F}\}.$$

Note that $L_{\mathcal{F}}$ is a rank s linear distribution on $\mathcal{X}$.

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Definition 2 An affine distribution $\mathcal{F}$ on an n-dimensional manifold $\mathcal{X}$ is *almost bracket-generating* if $\operatorname{rank}(\mathcal{F}^\infty) = n - 1$, and for each $x \in \mathcal{X}$ and any $\xi(x) \in \mathcal{F}_x$, $\operatorname{span}(\xi(x), (L_{\mathcal{F}^\infty})_x) = T_x\mathcal{X}$.

We also impose an additional condition in the definition of constant type—namely, that $\mathcal{F}$ is properly affine, and that each element $\mathcal{F}^i$ of the derived series is either properly affine or a linear distribution. This condition is reflected in the second condition of the following definition.

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Definition 3 The affine distribution $\mathcal{F}$ has *constant type* if

- $n_i(x)$ is constant on $\mathcal{X}$ for all i; i.e., the growth vector of $\mathcal{F}$ is constant on $\mathcal{X}$, and
- for any section ξ of $\mathcal{F}$, $\dim(\text{span}(\xi(x), (L_{\mathcal{F}^i})_x))$ is constant on $\mathcal{X}$ for all i.

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Local invariants depending on arbitrary functions appear in lower dimensions than for linear distributions: the first such invariants arise in the case of rank 1 affine distributions on 3-dimensional manifolds.

Elkin defines two control-affine systems

$$\dot{x} = a_0(x) + \sum_{i=1}^{s} u^i a_i(x), \tag{9}$$

$$\dot{y} = b_0(y) + \sum_{i=1}^{s} v^i b_i(y) \tag{10}$$

on state spaces $\mathcal{X}, \mathcal{Y}$, respectively, to be *affine equivalent* if there exists a diffeomorphism $\psi : \mathcal{X} \to \mathcal{Y}$ such that any absolutely continuous curve $x(t)$ in $\mathcal{X}$ is a solution of (9) if and only if the curve $y(t) = \psi(x(t))$ is a solution of (10).

Such a diffeomorphism must have the property that

$$\psi_*(a_0(x)) = b_0(\psi(x)) + \sum_{j=1}^{s} \lambda_0^j(x) b_j(\psi(x)), \tag{11}$$

$$\psi_*(a_i(x)) = \sum_{j=1}^{s} \lambda_i^j(x) b_j(\psi(x)), \qquad 1 \leq i \leq s \tag{12}$$

for some functions λ_0^j, λ_i^j on $\mathcal{X}$. In terms of the associated affine distributions $\mathcal{F}_{\mathcal{X}}, \mathcal{F}_{\mathcal{Y}}$, this is equivalent to the statement that $\psi_*(\mathcal{F}_{\mathcal{X}}) = \mathcal{F}_{\mathcal{Y}}$.

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1. *If $\mathcal{F}$ is almost bracket-generating, then in a sufficiently small neighborhood of any point $x \in \mathcal{X}$, there exist local coordinates (x^1, x^2) such that*

$$\mathcal{F} = \frac{\partial}{\partial x^1} + span\left(\frac{\partial}{\partial x^2}\right).$$

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2. *If $\mathfrak{F}$ is bracket-generating and $L_{\mathfrak{F}^2}$ (which must have rank 2) is Frobenius, then in a sufficiently small neighborhood of any point $x \in \mathfrak{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \left(x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2}\right) + span\left(\frac{\partial}{\partial x^3}\right).$$

3. *If $\mathfrak{F}$ is bracket-generating and $L_{\mathfrak{F}^2}$ is not Frobenius, then in a sufficiently small neighborhood of any point $x \in \mathfrak{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \frac{\partial}{\partial x^1} + span\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right),$$

where H is an arbitrary function on $\mathfrak{X}$ satisfying $\frac{\partial H}{\partial x^1} \neq 0$.

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According to (11), the drift vector field $a_0(x)$ may be replaced by any vector field of the form

$$a_0(x) + \sum_{j=1}^{s} \lambda_0^j(x) a_j(x),$$

and the resulting control system will be affine equivalent to the original.

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This motivates the following definition:

Definition 4 A *point-affine distribution* on a manifold $\mathcal{X}$ is an affine distribution $\mathcal{F}$ on $\mathcal{X}$, together with a distinguished vector field $a_0 \in \mathcal{F}$. Two point-affine distributions

$$\mathcal{F}_{\mathcal{X}} = a_0 + \mathrm{span}\,(a_1, \ldots, a_s)\,, \qquad \mathcal{F}_{\mathcal{Y}} = b_0 + \mathrm{span}\,(b_1, \ldots, b_s)$$

on the manifolds $\mathcal{X}, \mathcal{Y}$ (corresponding to the control-affine systems (9) and (10), respectively) will be called *point-affine equivalent* if there exists a diffeomorphism $\psi : \mathcal{X} \to \mathcal{Y}$ such that

$$\psi_*(a_0(x)) = b_0(\psi(x)), \tag{13}$$

$$\psi_*(a_i(x)) = \sum_{j=1}^{s} \lambda_i^j(x) b_j(\psi(x)), \qquad 1 \leq i \leq s \tag{14}$$

for some functions λ_i^j on $\mathcal{X}$.

We use Cartan's method of equivalence to give a local classification for point-affine distributions up to point-affine equivalence in the following cases:

- $\dim(\mathcal{X}) = n$, $\operatorname{rank}(\mathcal{F}) = n - 1$,
- $\dim(\mathcal{X}) = 3$, $\operatorname{rank}(\mathcal{F}) = 1$.

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- $\dim(\mathcal{X}) = n$, $\operatorname{rank}(\mathcal{F}) = n - 1$,
- $\dim(\mathcal{X}) = 3$, $\operatorname{rank}(\mathcal{F}) = 1$.

In all cases—even in the smallest nontrivial case of $\dim(\mathcal{X}) = 2$, $\operatorname{rank}(\mathcal{F}) = 1$—we find local invariants depending on arbitrary functions.

Theorem 6 *(CMW) Let $\mathcal{F}$ be a rank $(n-1)$ properly affine point-affine distribution of constant type on an n-dimensional manifold $\mathcal{X}$. Let $L_{\mathcal{F}}$ be the associated direction distribution of rank $(n-1)$, let v_1 denote the distinguished vector field in $\mathcal{F}$, and let η^1 be the unique 1-form on $\mathcal{X}$ satisfying*

$$L_{\mathcal{F}} = (\eta^1)^{\perp}, \qquad \eta^1(v_1) = 1.$$

Let k denote the Pfaff rank of η^1, i.e., the unique integer such that

$$\eta^1 \wedge (d\eta^1)^k \neq 0, \qquad \eta^1 \wedge (d\eta^1)^{k+1} = 0.$$

1. *If* $(d\eta^1)^{k+1} = 0$, *then in a sufficiently small neighborhood of any point* $x \in \mathcal{X}$ *on which* k *is constant, there exist local coordinates* $(x^1, \ldots, x^n)$ *and functions* $J_2, \ldots, J_{2k+1}$ *on* $\mathcal{X}$ *such that*

$$\mathcal{F} = \left(1 + \sum_{r=1}^{k} x^{k+r+1} J_{k+r+1}\right) \frac{\partial}{\partial x^1} + \sum_{r=1}^{k} \left(J_{k+r+1} \frac{\partial}{\partial x^{r+1}} - J_{r+1} \frac{\partial}{\partial x^{k+r+1}}\right) + \mathit{span}\left(\left(\frac{\partial}{\partial x^2} + x^{k+2} \frac{\partial}{\partial x^1}\right), \ldots, \left(\frac{\partial}{\partial x^{k+1}} + x^{2k+1} \frac{\partial}{\partial x^1}\right), \frac{\partial}{\partial x^{k+2}}, \ldots, \frac{\partial}{\partial x^n}\right).$$

2. *If* $(d\eta^1)^{k+1} \neq 0$*, then in a sufficiently small neighborhood of any point* $x \in \mathcal{X}$ *on which* k *is constant, there exist local coordinates* $(x^1, \ldots, x^n)$ *and functions* $J_1, \ldots, J_{2k+1}$ *on* $\mathcal{X}$ *such that*

$$\begin{aligned}
\mathcal{F} =& \Big(x^{2k+2} + \sum_{r=1}^{k} x^{k+r+1} J_{k+r+1}\Big)\frac{\partial}{\partial x^1} \\
&+ \sum_{r=1}^{k}\left(J_{k+r+1}\frac{\partial}{\partial x^{r+1}} - J_{r+1}\frac{\partial}{\partial x^{k+r+1}}\right) - J_1\frac{\partial}{\partial x^{2k+2}} \\
&+ \textit{span}\left(\left(x^{2k+2}\left(\frac{\partial}{\partial x^2} + x^{k+2}\frac{\partial}{\partial x^1}\right) - J_2\frac{\partial}{\partial x^{2k+2}}\right), \ldots,\right. \\
&\left(x^{2k+2}\left(\frac{\partial}{\partial x^{k+1}} + x^{2k+1}\frac{\partial}{\partial x^1}\right) - J_{k+1}\frac{\partial}{\partial x^{2k+2}}\right), \\
&\left(\frac{\partial}{\partial x^{k+2}} - J_{k+2}\frac{\partial}{\partial x^{2k+2}}\right), \ldots, \\
&\left.\left(\frac{\partial}{\partial x^{2k+1}} - J_{2k+1}\frac{\partial}{\partial x^{2k+2}}\right), \frac{\partial}{\partial x^{2k+2}}, \ldots, \frac{\partial}{\partial x^n}\right).
\end{aligned}$$

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where J is an arbitrary function on $\mathfrak{X}$.

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$$\mathcal{F} = \left(x^2 \frac{\partial}{\partial x^1} - J_1 \frac{\partial}{\partial x^2}\right) + span\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right),$$

where J_1 is an arbitrary function on $\mathcal{X}$.

3. *If $\mathfrak{F}$ is bracket-generating and $L_{\mathfrak{F}}$ is a contact distribution, then in a sufficiently small neighborhood of any point $x \in \mathfrak{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \left((1 + x^3 J_3)\frac{\partial}{\partial x^1} + J_3 \frac{\partial}{\partial x^2} - J_2 \frac{\partial}{\partial x^3}\right) + span\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right),$$

where J_2, J_3 are arbitrary functions on $\mathfrak{X}$.

Theorem 7 *(CMW) Let* $\mathfrak{F}$ *be a rank 1 properly affine point-affine distribution of constant type on a 3-dimensional manifold* $\mathfrak{X}$*.*

Theorem 7 *(CMW) Let $\mathfrak{F}$ be a rank 1 properly affine point-affine distribution of constant type on a 3-dimensional manifold $\mathfrak{X}$.*

1. *If $\mathfrak{F}$ is almost bracket-generating, then in a sufficiently small neighborhood of any point $x \in \mathfrak{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \left(\frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3} \right) + span \left(\frac{\partial}{\partial x^3} \right),$$

where J is an arbitrary function on $\mathfrak{X}$.

2. *If $\mathfrak{F}$ is bracket-generating and $L_{\mathfrak{F}^2}$ (which must have rank 2) is Frobenius, then in a sufficiently small neighborhood of any point $x \in \mathfrak{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \left(x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3} \right) + span \left(\frac{\partial}{\partial x^3} \right),$$

where J is an arbitrary function on $\mathfrak{X}$.

3. *If* $\mathcal{F}$ *is bracket-generating and* $L_{\mathcal{F}^2}$ *is not Frobenius, then in a sufficiently small neighborhood of any point* $x \in \mathcal{X}$*, there exist local coordinates* (x^1, x^2, x^3) *such that*

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1} + J\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)\right) + \mathit{span}\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right),$$

where H, J *are arbitrary functions on* $\mathcal{X}$ *satisfying* $\frac{\partial H}{\partial x^1} \neq 0$*.*

3. *If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is not Frobenius, then in a sufficiently small neighborhood of any point $x \in \mathcal{X}$, there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1} + J \left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3} \right) \right) + \mathit{span} \left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3} \right),$$

where H, J are arbitrary functions on $\mathcal{X}$ satisfying $\frac{\partial H}{\partial x^1} \neq 0$.

Compare Theorem 7 with Theorem 5; in each case there is a new invariant J which reflects the restriction that v_1 is a fixed vector field in $\mathcal{F}$.

Example 3 (Navigation on a river) Consider a boat navigating on a river with a current; for simplicity, assume that the current runs parallel to the x-axis with constant speed c. The state of the boat is represented by its position (x, y) and heading angle ψ (see Figure 1); thus the state space is $\mathcal{X} = \mathbb{R}^2 \times S^1$.

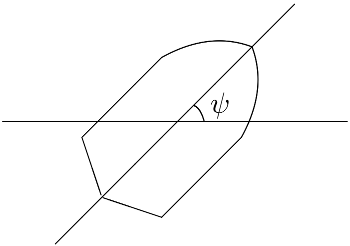

Figure: Heading angle ψ in Example 3

Due to asymmetry in the shape of the boat's hull, the current may have a rotational effect as well as a translational effect on the boat, and this effect depends on the boat's heading angle. Thus we will assume that the drift vector field has the form

$$v_1 = c\frac{\partial}{\partial x} + r(\psi)\frac{\partial}{\partial \psi}.$$

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$$v_1 = c\frac{\partial}{\partial x} + r(\psi)\frac{\partial}{\partial \psi}.$$

We also assume that the boat can be propelled in a forward or backward direction, corresponding to the vector field

$$v_2 = (\cos\psi)\frac{\partial}{\partial x} + (\sin\psi)\frac{\partial}{\partial y},$$

and that it can be steered to the right or left, corresponding to the vector field

$$v_3 = \frac{\partial}{\partial \psi}.$$

The navigation problem for the boat is then the control-affine system corresponding to the rank 2 point-affine distribution

$$\mathcal{F} = v_1 + \mathrm{span}(v_2, v_3)$$

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Note that $\mathcal{F}$ fails to be properly affine precisely when $\psi = 0$ or $\psi = \pi$, since then v_1 is contained in $L_{\mathcal{F}}$. Thus our results are only applicable when the boat is *not* pointed directly upstream or downstream. Since $L_{\mathcal{F}} = \mathrm{span}(v_2, v_3)$ is a contact distribution, this system falls into Case 3 of Corollary 2.

The local coordinates of Corollary 2 are:

$$x^1 = \frac{x}{c}, \qquad x^2 = \frac{y}{c}, \qquad x^3 = \cot\psi.$$

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In terms of these coordinates, we have

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1} - ((x^3)^2 + 1) r(\cot^{-1}(x^3)) \frac{\partial}{\partial x^3} \right) + \operatorname{span}\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right).$$

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Thus the local invariants are:

$$J_2 = ((x^3)^2 + 1) r(\cot^{-1}(x^3)) = (\csc^2\psi) r(\psi), \qquad J_3 = 0.$$

Questions for further study:

- What metric structures (analogous to sub-Riemannian or sub-Finsler geometry for linear distributions) are appropriate for point-affine distributions, and what can we say about their geometry, geodesics, etc.?

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- What metric structures (analogous to sub-Riemannian or sub-Finsler geometry for linear distributions) are appropriate for point-affine distributions, and what can we say about their geometry, geodesics, etc.?

  Even for linear distributions, issues such as controllability and the presence of *rigid curves*—i.e., curves with no C^1 variations whatsoever among horizontal curves—are nontrivial, and the study of optimal trajectories is quite complicated. These issues are even less well-understood for affine distributions, but they are obviously important for understanding the associated control problems.

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  This issue is crucial for systems arising in mechanics, such as that in Example 2:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_i(x) \end{bmatrix} \begin{bmatrix} u^i \end{bmatrix}.$$

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  The associated affine distribution fails to be properly affine precisely along the submanifold $\{p = 0\}$ of stationary points.

  The behavior of trajectories near such points is of vital interest in control theory, so it will be necessary to study the geometry of such structures. Elkin has introduced the notion of a "t-codistribution," and we anticipate that this will be a useful tool for extending our methods to affine distributions of this type.