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Mathematics Education Dialogues

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ALGEBRA? A GATE! A BARRIER! A MYSTERY!

In increasing numbers, policymakers are calling for algebra for all. Many states and school districts have mandated it either in specific grades or by graduation. Colleges and universities demand it for admission without remediation. Some call it a civil right and base a part of the argument on economics. But what is it? What is "algebra," and who is "all"?

For some time now, the National Council of Teachers of Mathematics, the Mathematical Sciences Education Board, and other organizations have been thinking about the nature and role of algebra in the K–14 curriculum. These groups have struggled with the meaning of *algebra* and that of *all*. What algebraic topics are many universities demanding on their placement tests? Does implementing "algebra for all" dumb down the curriculum for college-bound students, as some claim? Is it really possible for all students, even those who have been tracked out of a college-intending curriculum, to learn algebra? If so, what are the big ideas that all should know? Are the topics being demanded relevant, or has technology made some of the topics inherently irrelevant? Is algebra itself an artificial cultural barrier created in another era but totally unnecessary in today's world for a large segment of the population? Is algebra really a civil right? What are problems of implementing algebra in middle schools? Is the groundwork necessary for algebra being laid in early grades? And finally, what changes should be made in teacher preparation if algebra is to be taught in grades K–14?

This issue of *Mathematics Education Dialogues* is devoted to beginning a wider discussion about algebra, with an emphasis on practical concerns. Articles by teachers at different levels and with different expectations are presented. Algebra for all is a multifaceted issue. Indeed, few people have no opinion on what, how, or why? What is your position?

by Johnny W. Lott, Editor
For the Editorial Panel

Inside this issue . . .

Algebra for All? Why?

by Nel Noddings

Many secondary schools require all students to pass courses—and sometimes, standardized tests—in algebra. One argument for this requirement focuses on equality of opportunity; another argues that algebra is widely needed in the world of work. Both arguments are questionable. My objections are here directed at the practice of requiring the traditional algebra courses, not at teaching specific topics selected wisely from algebra.

The argument that focuses on equality says, roughly, that all children should have the educational opportunities once reserved for a few. It's hard to oppose such a generous gesture, but some points should be raised in opposition. First, the requirement begs fundamental educational questions: Why is academic mathematics a requirement for college entrance? Why is it not possible for American students, as it is for students in many other parts of the world, to choose a humanities track for college preparation? Second, can we claim to be offering an opportunity through coercion? Because of lack of adequate preparation or lack of interest, many students will fail algebra; and with the advent of high-stakes testing, many students who could have earned a high school diploma will fail to get one.

In our zeal to give everyone a chance at college, we are tolerating—perhaps even encouraging—courses that bear little resemblance to "real" algebra. Youngsters achieve a grade on their transcripts and then find that they are utterly unprepared for college-level work. This practice borders on fraud and scarcely represents "equal opportunity."

In most parts of the United States, treatment of non-college-bound students is shameful. Instead of providing carefully planned, interesting, and honestly advertised courses for such students, we force everyone into courses suitable for only some future career-option choices. Instead of showing truly democratic respect for all honest work and for the students who will someday do this work, we treat everyone as college-bound. Students who take jobs without going to college do so by default because they are considered "not good enough" for college.

But maybe everyone needs more mathematics today. Perhaps the requirement has a genuine educational rationale. I doubt it. Although some relatively new occupations surely use lots of mathematics, many older ones use less than they did fifty years ago. Mathematics may underlie much of what is done in such occupations, but the workers themselves need very little. Arguments for the nonoccupational uses of mathematics—citizenship, personal finance, and so on—are similarly flawed. They might provide a rationale for overhauling the mathematics curriculum, but they cannot support requiring traditional courses in algebra.

A genuinely useful course for many students in high school mathematics would include, in addition to unquestionably useful mathematical skills, something about the politics of mathematics education: what it means to live in a mathematicized world, how mathematics has been used as a gatekeeper, the difference between knowing a subject and having a credential, how tests discriminate both fairly and unfairly, how tests are constructed, how students contribute to their own lower economic status by unreflective resistance to mathematics courses, and what intelligent resistance might look like.

Students should be able to choose either college or noncollege curricula proudly and with some confidence that their choices will yield a genuine valued education.

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The purpose of *Mathematics Education Dialogues* is to provide a forum through which NCTM members can be well informed about compelling, complex, timely issues that transcend grade levels in mathematics education. The opinions expressed in this publication are those of the writers and do not constitute an official position of the NCTM. *Mathematics Education Dialogues* is published as a supplement to the *News Bulletin* by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-9988. Pages may be reproduced for classroom use without permission.

ALGEBRA FOR ALL: *It's a Matter of Equity, Expectations, and Effectiveness*

by Dorothy S. Strong
and Nell B. Cobb

Any discussion of the appropriate role and importance of algebra in the school mathematics curriculum eventually boils down to answering four basic questions:

- If not algebra, then what?
- If not for all children, then for whom?
- If not at all schools, then in which ones?
- If not now, then when?

In our minds, answers to these questions are fundamentally related to issues of equity, expectations, and effectiveness.

Algebra, which Robert Moses (1994) and others consider to be a civil right for all Americans, has always been a civil right for a select few. Moses insists that the struggle for equality for minority people is directly linked to mathematics and scientific literacy. We believe that the successful completion of algebra will ensure that "all students are prepared for college level mathematics or mathematics related careers" (Strong 1999, 3). We envision all schools being transformed into places where all students successfully complete prealgebra by seventh grade, a full year of algebra by eighth grade, and four years of high school mathematics.

For this vision to become a reality, students, parents, teachers, and administrators—indeed, all members of society—must believe and expect that all children can and must learn algebra. They must aggressively pursue—both individually and collectively—their responsibilities to this vision. Only when this commitment happens will all students learn algebra and thereby break the illiteracy cycle in mathematics and science that has reached epidemic proportions in some schools.

Mastery of algebra must also be a K–12 undertaking. That is, the study and use of algebra must begin at the start of every student's formal schooling and must continue throughout each student's academic experience. In addition, truly effective algebra instruction must be both historically accurate and culturally relevant. For example, students should learn that "the word *algebra* is Arabic in origin and that Europe received algebra as a gift from Africa and Asia" (Chicago Public Schools 1990, 48).

Teacher preparation and subsequent professional development raise equity and expectation issues. To minimize the impact of students' mathematical deficiencies and to enable disadvantaged students to compete on the same level playing field as advantaged students, teachers must be prepared to remediate students' deficiencies while helping them learn new concepts and procedures. To accomplish this challenging task, technological support is critically important. Also, we have found that teachers should be prepared to teach by following an approach that consolidates selected principles of both standards-based and traditional mathematics instruction. This approach to instruction, called *BiMathematics,* is similar to bilingual instruction in that it produces students who are proficient in the two languages—skills-based and standards-based—of mathematics.

To sum up, we firmly believe that all children, at all schools, can have algebra competence now. With nothing cast in stone, we must assess the validity of everything that is done in the name of mathematics education and create school mathematics programs that embrace equity, expectations, and effectiveness.

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Dorothy S. Strong

Nell B. Cobb

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ELEMENTARY Grades

Don't Delay:

Build and Talk about Rich Experiences from the Beginning

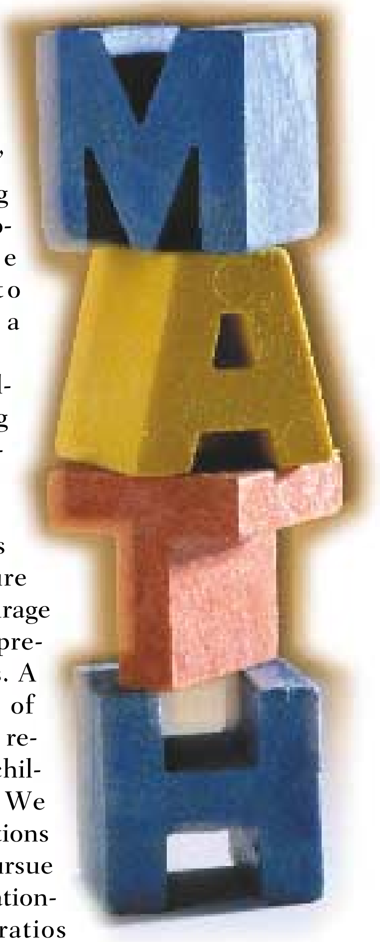
by Sydney Schwartz and David Whitin

As we ponder the wisdom and implications of *algebra with delayed symbolism in the elementary school*, we face the question of what we do instead. The idea of delaying the use of symbols assumes that children will develop the necessary level of abstract thinking without direct instruction. Do we sufficiently understand how symbolic thinking in algebra develops so that we can recognize and support the process? If so, how do we prepare primary teachers to do this?

In the February 1997 focus issue of *Teaching Children Mathematics,* contributors discussed both observed and presumed ways that young children develop algebraic thinking. With an emphasis essentially on algebraic reasoning, articles illustrated strategies for engaging children in explaining ideas in graphic forms, primarily pictorial symbols. Our comprehension of the process by which a child structures understandings of this content is essential to selecting teaching strategies that foster progress from a single representation of a relationship to a generalization.

When we asked 25 teachers with master's degrees to define algebra, the preponderance of answers focused on figuring out a value of *x* or a missing number. To support children's translation of thinking into symbolic representations about numerical relationships, patterns, and generalizations, a more comprehensive understanding of algebra is needed. This limited view of algebra is not sufficient if we are to modify our instructional approach to follow the children's lead rather than to lead them. The current educational emphasis on the child's construction of knowledge, illustrated in the focus issue, highlights the importance of the teacher's knowledge base to successfully engage children in using their emerging algebraic reasoning and illustrating it graphically or symbolically.

For example, when one kindergarten child was observed constructing trains of blocks using pairs of colors in single alternation, further "kid watching" revealed that the child was developing an understanding of the *abab* relationship. She commented that her two trains were the same because "they kept moving back and forth." Multiple experiences creating visual, tactile, auditory, and motoric *ab* patterns provide the context for the idea to grow on the path to becoming a generalization.



If we are to consider a developmental approach to teaching algebra with the timely inclusion of symbols, then we need to prepare teachers not only to recognize algebraic thinking as it emerges but also to structure authentic situations that encourage children to use symbols to represent patterns and relationships. A consideration of delayed use of symbolism in teaching algebra requires knowing how to foster children's emerging symbolism. We need to go beyond the explorations that children spontaneously pursue in such areas as functional relationships in geometry, common ratios across nonstandard units in measurement, transformations of computational problems into convenient numbers, and numerical equalities and inequalities; we need to create everyday situations that engage children in algebraic reasoning and representations. To these authors, the operant word is not *wait* but rather *scaffold,* that is, stimulate the building of bridges from the concrete to the abstract.

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ELEMENTARY Grades

Exploring Our World through Algebraic Thinking

by Barbara Moses

Rachel, a fourth grader, noticed that her brother always ate more pizza than she did. When Rachel wasn't very hungry, she only ate one slice, and Daniel ate three. When Rachel was very hungry, she ate three slices, but Daniel ate five. And when she was very, very hungry, she ate four slices, and Daniel ate six. Rachel noted, "He always eats two more slices than I do. If I eat six slices, I bet he'll eat eight slices." When asked to give a number sentence to describe these circumstances, Rachel wrote the following:

Number of slices Rachel eats + 2 = number of slices Daniel eats

Rachel was engaging in algebraic thinking.

Algebra is all about identifying patterns and expressing these patterns in some fashion. It has been described as mathematics that deals with general statements of relations, using letters and other symbols to represent specific sets of numbers, and so on. To do algebra, students must learn to collect data and organize those data, to recognize patterns in the data, and to express the patterns that they have observed. If students can reason in this manner, we have helped them think mathematically and have bridged the gap from arithmetic to algebra.

Young children are not only capable of noticing patterns but often use this skill naturally to make sense of their world. They achieve remarkable results on their own when the situation is open-ended. For example, when a teacher in one classroom asked students to create rectangles with an even number of colored tiles, they began with two tiles and created a two-by-one rectangular object. They continued to create rectangles with four tiles, constructing both two-by-two and four-by-one objects; with six tiles, they created two-by-three and six-by-one objects; and so on. They noticed that when they used an even number of tiles, one dimension could always be two. They concluded that every even number could be written as two times some other number—a small step away from an even number being $2x$. They examined an odd number to see what would happen if one dimension was two. Because there was always an extra tile, they decided that an odd number is an even number plus one—very close to writing an odd number as $2x + 1$.

As technology plays a more significant role in curricula, the emphasis in algebra is shifting. No longer is algebra solely focused on such rote learning of symbol manipulation as solving equations and simplifying expressions. Calculators can now factor expressions and solve equations. So why does algebra remain an integral ingredient of the mathematics curriculum?

Algebra is essential because it is a way of thinking, a way of interpreting and understanding situations in daily life. The world is full of regularities that students can identify. The money that they make on their paper routes increases as the number of papers delivered increases; the number of cookies needed for a class celebration depends on the number of students in the class. But the money that they make and the cookies that they need are variables related to other variables: number of delivered newspapers and number of classmates, respectively. Interpreting these relationships, modeling them, and predicting future events become foci of algebra.

Algebraic thinking is the vehicle for exploring the world and its regularities. A young child may investigate the table of basic addition facts and note symmetry across a diagonal, thus recognizing the commutative property. An older child may note that the sum of the first n odd natural numbers seems to generate square numbers. Another child may use spreadsheets to investigate exponential growth. As we study patterns in our world, we develop an awareness of regularities and find ways to represent them. The representations of these relationships may change according to the maturity of the students, but we must encourage algebraic thinking at every level.

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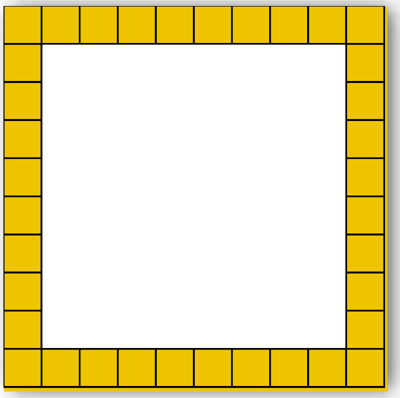
Algebra in the Middle School? YES, and Here's How to Start!

by Angela F. Allen

I am a proponent of algebra in the middle school. If appropriately taught, algebra helps students reason, see patterns and relationships, and make predictions and generalizations. That is, it engages students in active learning that is at the heart of what mathematics is all about. The key is to teach algebra appropriately!

If algebra is presented to students as a collection of algorithms and procedures to memorize without any connections to underlying meaning, then algebra is not appropriate for middle school students. But if it is taught in a way that involves students in searching for patterns and relationships and in making generalizations that are based on sound reasoning, it can be an intellectually stimulating and productive opportunity.

All students in my school took a full year of algebra in eighth grade. In reflecting on my experience teaching those students, I typically selected tasks having three important characteristics: legitimacy for the students, alignment with students' mathematical thinking, and inclusion of important mathematical concepts or processes. For example, traditionally at the start of a course, I used the border problem from *A Collection of Math Lessons from Grades 6 through 8*, by Marilyn Burns and Cathy McLaughlin (New York: The Math Solutions Publications, 1990). This problem requires students to find the number of 1 × 1 squares on the border of a 10 × 10 grid. In a classroom environment where students felt free to express their ideas and methods, they assumed ownership of the task as they created and shared their methods for finding the unknown—the number of squares contained in the border.



Students used several methods for counting the squares on the border. For example, some students found the number of squares on one side of the border (10 squares); multiplied by 4; and then subtracted 4, recognizing that the corners were counted twice, to arrive at 36 squares. They tested the generalizations of the methods on grids of other sizes. After they arrived at a correct generalization, I asked them to predict and to test to their generalizations on a 100 × 100 grid. Some students complained about the time needed to count that many squares.

I then asked the students to translate their methods into algebraic language to describe their procedures. I suggested that they start by changing the words in their descriptions to letters or variables, explaining that the unknown values would change for different squares.

After some confusion and consternation, students translated their counting methods into algebraic language. For example, students who used the previously described method substituted the letters B and S for the number of squares on the border and on the side, respectively, and wrote $S \times 4 - 4 = B$. This task clearly does not require high-level cognitive demands or mastery of basic facts, yet it helped the students see how algebra is a natural extension of arithmetic.

This task, as well as others used with my students, changed the way that they viewed mathematics. They saw that algebra in middle school was not a muddle but was a way to think about and do mathematics that involved finding patterns and making generalizations. More significantly, they were excited to learn that mathematics was more than the same arithmetic that they had done in sixth and seventh grades. Through the use of good tasks and a supportive classroom environment, my students gained mathematical confidence and saw how algebra connects to their previous learning experiences, how it offers a new way to approach and solve problems, and how it establishes the foundation for higher-level mathematics.

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by Tami Matsumoto

Demanding More from Middle-Level Mathematics?

Secretary of Education Richard Riley (Riley 1998) implied that all students should complete algebra and geometry by the end of the ninth grade. For that effort to be successful, our typical approach to those courses needs rethinking. Middle-level students are not ready for algebra and geometry courses as they are traditionally taught in high school. In this issue of *Dialogues*, Sid Rachlin cites evidence that these traditional courses may not be working even at the high school level. Merely dropping the same courses to lower grade levels would be similar to dropping a brick wall in front of most eighth and ninth graders.

Eighth and ninth graders are capable of solving complex problems and doing rigorous mathematics with high-level thinking—but we must be careful how we approach teaching abstract concepts. We need to provide bridges from students' concrete experiences, as suggested in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

Success in formal high school algebra and geometry courses requires more than arithmetic skill and hard work: it requires an ability to reason abstractly. That ability is something that only a small percentage of eighth graders have developed (Jensen 1998). Most 12-to-15-year-olds have not yet developed the capacity for abstraction at the levels required by these courses as they are typically configured.

Our most capable mathematics students usually take algebra and geometry courses in the eighth and ninth grades. These students have proven arithmetic skills, an interest in mathematics, and a willingness to work at it. Some of them struggle, some struggle and fail, and some eventually quit taking mathematics, even though they are still highly capable mathematically.

Those students who take these courses ahead of the typical schedule are not the only ones who encounter difficulties. Large numbers of ninth- and tenth-grade students also fail formal algebra and geometry courses. I do not believe that all who fail are simply not working hard enough or that they are incapable of understanding the concepts. Rather, they are not ready for the kinds of thinking required of them in the coursework in its present form.

Both groups of students are victims of a system that is not tailored for their mental growth and learning needs. Teaching the current formal courses in middle grades is not a good proposal. We can and should demand more of middle-level students, but our approach must be different.

First, we can work in the earlier grades to better equip students for later formal algebra and geometry studies. Classrooms in elementary and middle schools that furnish the curriculum, instruction, and assessment envisioned in the *Standards* will help prepare students to understand concepts of algebra and geometry. Students who encounter algebraic and geometric ideas in concrete forms and real-world contexts in early years and whose learning is conceptually based will better make connections from prior experiences to more abstract ideas. They should be more prepared to understand algebraic ideas than their traditionally trained counterparts who see those ideas for the first time in the abstract in a formal high school course.

Equipping students better is only part of the solution. We must rethink the system at the high school level. Instead of courses designed for abstract thinkers, which many of our students are not, let's redesign the coursework to consider what we have learned about cognitive development and learning styles. We can offer instruction in algebra and geometry that bridges the concrete and the abstract and that builds on students' elementary and middle-level algebraic and geometric experiences.

We can meet our goal of mathematical power for all students without pushing high school courses into the middle level. We cannot and should not expect all students to succeed at the pace and abstraction levels of the traditional courses.

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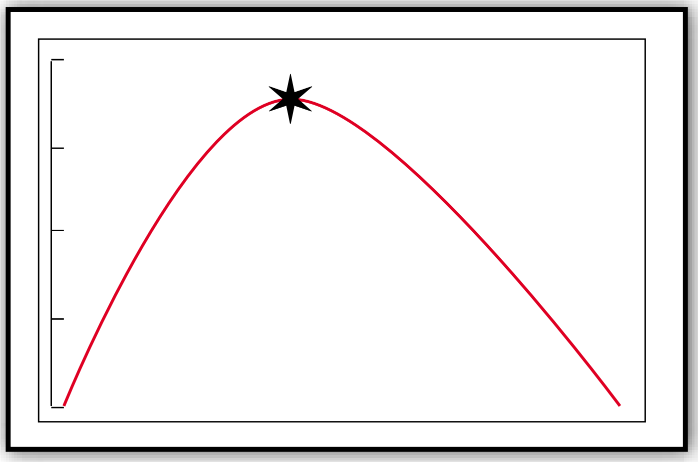
Making High School Algebra Accessible to All

by Kelly Hodges

The mandate arrives from administration: passing algebra is now a graduation requirement. The requirement is well-intentioned. By allowing only certain students access to algebra, schools perpetuate an inequitable setting that runs counter to the schools' purpose. But teachers gasp, "What about those who aren't going to college? What about those with learning disabilities?" Merely placing all students in traditional algebra classes will not mend the rift of inequity. The typical algebra course was never designed to meet the needs of "all" students.

When trying to design an algebra course for all students, my colleagues and I began by clarifying the goals of such a course. It should expand a student's mathematical frame for viewing the world. Arithmetic works well for describing static pictures of the world, but mathematics can do more than that. The course should leave students prepared to continue their mathematical studies. But topics cannot be included solely to prepare for the next class. Finally, the structure of such a course should help students learn how to ask and answer their own mathematical questions.



We have had some success in meeting these goals with a course that we designed around the concept of function. In this introductory course, a variable represents a changing quantity rather than an unknown number, and a function is a relationship between two variables. Students study multiple representations of functions, especially linear functions, and focus on their relative strengths and weaknesses in solving different types of problems. Solving equations is a focus, but it comes late in the course, and symbol manipulation is only one of many possible strategies considered. The feeling of a multitude of disconnected facts and procedures common to other algebra courses is eliminated. The function perspective for the course allows for a shift from static to dynamic in a way that a "generalized arithmetic" perspective on algebra does not. Our new course has served students well in their future classes, but it has also been accessible and engaging to students who would never have passed a traditional algebra course.

Integrating technology into daily instruction makes deeper concepts accessible. In addition, it helps provide equitable access to students with computational learning disabilities as well as those with visual or physical disabilities that impair their ability to write or draw. For our students, functions become mathematics that they see in the world around them; functions are powerful ways of modeling situations that change. Although the function orientation of the course has provided a perspective that allows us to address the goals of algebra for all, the pedagogy that has developed along with the curriculum has been just as important.

We have found that ideas should first be rooted in real-world contexts to allow an abstract idea to have a metaphor in a student's experience. Instruction should then grow from student questions to ensure that topics connect to one another in the minds of the students. We have learned to design activities that both elicit student questions and support generalization, so that the skills gleaned from one context are shaped and restated in a widely applicable form. We assess students on their abilities to apply new skills and generalizations in different contexts than those from which they arose. In watching students bring concepts to new and unfamiliar situations, we see what they have learned.

As a society, we are obligated by the ideals of democracy to offer to all students an algebra course that is rigorous, important, and accessible. Traditional algebra classes have been unsuccessful in this regard. Our functions-based approach has led to the development of the described course that meets these demands.

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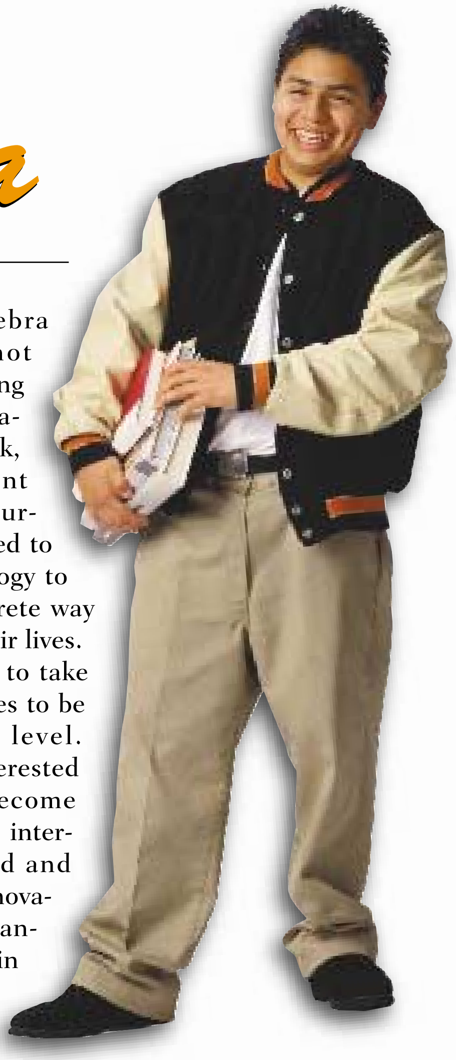
HIGH School

No! Not Algebra

by Shirley T. Bagwell



One does not need to know how to build a car to drive one, nor does one need to be able to program a computer to use one. Much software is being developed that helps people manage their lives and solve problems. Anyone who uses a calculator, a spreadsheet, measuring devices, or other technology can solve real-life problems without knowing algebra. Some algebra seeps into our lives in the form of formulas. If this occurs naturally, even the most reluctant student will accept and use the formulas. However, to expect all students to have an interest in, or aptitude for, algebra is to create an atmosphere of failure and frustration for some students.

Many students cannot grasp the abstract nature of algebra topics. They quickly become frustrated and tune out all attempts to help them. Some students become discipline problems. They are not academically ready to make the step from concrete to abstract. Special education students are encouraged to take courses in the mainstream curriculum. For example, after making slow progress in arithmetic, Sue began to understand the meaning of numbers in junior high school. She learned to solve concrete problems using a calculator. When placed in an algebra-based course in high school, she gave up. Even with help from the teacher and a tutor, she could not grasp equations. If students like Sue are placed in an algebra class, they quickly feel inadequate and quit trying. If they are taught to use technology to solve problems that they encounter daily, they grow in understanding mathematics.

Algebra is not for everyone, just as college is not for everyone. We do not require all students to take chemistry, physics, or a foreign language to graduate from high school. If students who have not mastered prealgebra skills are required to take algebra, many will fail. Failure does nothing for the slower student but lower self-esteem and reaffirm a belief in inability to learn. To expect all students to master algebra objectives is unreasonable and will cause more students to drop out of high school. For example, Jack, an average student in most classes, can do arithmetic using a calculator, but he needs extra help with problems that require two or more steps. He repeated algebra in his junior year because it was a graduation requirement. He never developed the skill to "translate" from English to algebra, and he never understood equations. He became discouraged and escaped class by sleeping through it. No amount of encouragement could keep him awake.

Everyone needs arithmetic to solve problems and should acquire the mathematical skills to understand, organize, and solve the problems that they encounter daily. Many students are pushed into an algebra course when they do not know how to use measuring devices and units of measure, balance a checkbook, apply for a loan, or count change after making a purchase. These students need to learn how to use technology to solve problems in a concrete way that applies directly to their lives.

Requiring all students to take algebra forces those classes to be taught at a more basic level. While the weaker, disinterested students struggle and become discouraged, the stronger, interested students are bored and restless. Even the most innovative teaching strategies cannot bridge so wide a gap in student performance and interest. Some students who are not interested in mathematics live their lives without seeing a need for algebra. Others find applications in many areas. Mathematics students who memorized and bluffed their way through required algebra courses with barely a passing grade will avoid mathematics forever and remember the subject as a nightmare. If they are encouraged to use technology to explore problems that interest them, they may later want to study algebra.

To think that all students are ready to study algebra in high school is naive and certainly contrasts with the reality of our society, in which students have varied interests and ability levels. We must consider a student's strengths and interests when developing a course of study for that student. Although everyone needs the skills to solve problems, not everyone needs to use algebra to solve them. 🍎

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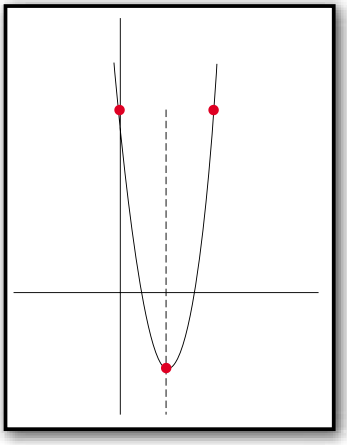
HIGH School and Beyond

Rising to the CAS Challenge

by Mark Howell

The advent of inexpensive handheld calculators equipped with robust computer-algebra systems (CAS)—such as the TI-89, HP49G, and Casio FX 2.0 confronts every mathematics educator with choices. Because more students are coming into classrooms toting machines that perform many of the ritualized tasks that are in every algebra course, we must ponder what algebra is all about and what we should be spending our time emphasizing. These new calculators, for example, will isolate a variable in an equation whenever it is possible and will factor every polynomial that a student may see in school mathematics. When graphing calculators came into mathematics classrooms, we had a chance to step back and reassess what we teach and why. Many teachers felt invigorated by that opportunity. The stakes seem higher when symbolic-manipulation skills are on the table. But the very same opportunity awaits us. How shall we respond?

Opponents of using CAS as a tool to teach algebra suggest that its use undermines students' paper-and-pencil skills. Significant evidence indicates that this need not be the case. M. Kathleen Heid showed that after a CAS-intensive algebra course, where a much shorter time is spent on paper-and-pencil skills, students' skills meet or exceed those of their counterparts who pass through a traditional course (Heid and Zbiek 1999). O'Callaghan (1998) showed that students who participated in a CAS-intensive course emerged with a stronger understanding of function and of the connections among their graphic, numeric, and symbolic representations than students from a more traditional course that did not incorporate technology.



A typical algebra example is to transform the equation $y = 2x^2 - 12x + 13$ into a form that is easily graphed and find its vertex and axis of symmetry. Examining such content, we should consider the algorithms that we teach to sort out the relevant mathematics. Here, is it in knowing how to manipulate $y = 2x^2 - 12x + 13$ into $y = 2(x^2 - 6x) + 13$, then complete the square by writing $y = 2(x^2 - 6x + 9) + 13 - 18 = 2(x - 3)^2 - 5$, and read off the vertex as (3, –5) and the line of symmetry as $x = 3$? Or is the relevant mathematics the fact that the original equation is equivalent to the final one and the reason that the vertex and line of symmetry can be read from the transformed equation?

Students who have experienced technology-enriched mathematics courses may be frustrated if technology is banned altogether or used minimally in college courses. Some Advanced Placement calculus teachers, whose students use graphing calculators or CAS, counsel their students to avoid frustration by avoiding colleges and universities that do not include technology as an integral part of the mathematics program. Technology offers an observation tower from which connections among graphic, numeric, and symbolic representations of concepts can be seen. The following are some questions to think about and to seek answers to:

- How much paper-and-pencil symbolic ability is enough?
- What happens to students who pass through a course sequence that uses CAS or other technology in subsequent courses that prohibit such use? What about the students who do not use CAS in lower courses but are confronted with later courses that require technological literacy?
- Which symbolic algebra skills can we cast aside, and which do we clutch to our hearts?
- How does CAS help us teach mathematics effectively in tandem with numeric and graphic viewers?
- How do we evaluate symbolic algorithms to condense important mathematics that they contain?

The answers are found in using CAS with students and in finding out what others are doing with CAS.

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Heid, M. Kathleen, and Rose Mary Zbiek. "Planning a Mathematics Program for the Future." Everyday Learning Corporation's Math Link. www.everyday learning .com/Pages/authdesk.html. 1999. World Wide Web.

O'Callaghan, Brian R. "Computer-Intensive Algebra and Students' Conceptual Knowledge of Functions." *Journal of Research in Mathematics Education* 29 (January 1998): 21–40.

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Algebra in an Integrated Approach to Mathematics

by Andy Begg

High school mathematics in the United States is typically taught as algebra and geometry in different years. In many other countries, it is structured differently. In New Zealand, high school mathematics used to be arithmetic, algebra, and geometry, with trigonometry and calculus in the later years. The topics were taught on different days. Students had to bring the right textbook, exercise book, and homework to class. This system slowly changed as authors repackaged chapters into "general mathematics" textbooks.

When new math was introduced, the alternating-topics approach in the textbooks continued. Curriculum developers who saw mathematics in an integrated way had an influence, as did economics; one textbook each year was cheaper than three. At this stage, some "fudging" of boundaries began to emerge with such topics as sets, vectors, and matrices, which were relevant in algebra, geometry, and arithmetic.

Now the curriculum has changed again. For elementary and high schools, there are six topic strands: number, measurement (including calculus), geometry (including trigonometry), algebra, and statistics; and one integrated "mathematical processes" strand of problem solving, reasoning, and communication. The topics are usually taught in units of one to four weeks' duration and often involve more than one strand.

Multistrand approaches abound. With algebra, an approach that generalizes from number might also involve patterns from measurement and geometry. Transformational geometry uses functions and matrices to link with algebra. Coordinate geometry can be used to link graphs, algebra, and calculus. Area measurement can be used to link formulas, solving equations, and ratios of areas of shapes when different units of measure are involved. Alternatively, the emphasis could be geometric, involving breaking complex shapes into simpler ones; or trigonometric, if different information was given about the shapes. Area is also viewed algebraically as a function that maps shapes onto numbers. Statistics about area data of houses, building lots, farms, cities, and so on, combine at least two areas of mathematics.

I see four advantages of integrated mathematics:

- All students from age five learn algebra along with mathematics from the other strands; students do not have a year in which they forget what they have learned.
- Integrated courses facilitate making connections, and algebra becomes involved in a range of contexts.
- The integrated curriculum provides variety, which is important because students enjoy parts of mathematics but lack confidence with others.
- An integrated philosophy enables changes to be made at each level so that such new topics as statistics can be introduced easily.

Andy Begg, a.begg@waikato.ac.nz, teaches mathematics education at the University of Waikato in New Zealand. He was previously a high school mathematics teacher, a school principal, a coauthor of high school textbooks, and a government curriculum officer.

The Algebra That Students Need to Know to Avoid Remedial Mathematics Courses in College

by Sid Rachlin

One of the first experiences that a new student encounters upon entering a university is taking a placement test for mathematics. These tests furnish one measure of the algebra that students have to know from kindergarten through twelfth grade if they are to begin postsecondary education.

On the basis of the results of these tests, many high school graduates seem to lack necessary algebraic skills and to require remediation. Any dialogue on the nature of K–12 algebra would be incomplete if it did not consider the algebraic skills that universities expect students to bring with them for postsecondary work.

During 1998–1999, the University of North Carolina (UNC) system spent more than two million dollars on remediation for 6369 students. Of these students, 74 percent were specifically enrolled in remedial mathematics courses. In an effort to make high school students more aware of the algebraic skills that colleges expect and to provide the students a reality check of their current skills, the general administration of the UNC system sponsors the North Carolina Early Mathematics Placement Testing (EMPT) program. EMPT provides interested high schools with a version of the Mathematical Association of America's Algebra Placement Test. Last year, 27 030 students completed the 32-item multiple-choice test. Of these students, 69 percent were completing second-year algebra and the remainder were completing a course beyond it. After graduating from high school, in one or two years, 83 percent of the students planned to enter a four-year college and 13 percent planned to attend a two-year college.

The following test items offer a sample of the types of questions asked. (Calculator use was optional.) As shown in the item analysis in **table 1,** these two items were answered correctly by 39 percent and 36 percent of the students, respectively.

- If $f(x) = 2^x + x^2$, then $f(-1) =$

  (A) $^3/_2$ (B) $^1/_2$ (C) $^{-1}/_2$ (D) -1 (E) $^{-3}/_2$

- An equation of a line passing through $(-3, 6)$ with slope $^{-4}/_5$ is

  (A) $4x + 5y = -18$ (B) $4x + 5y = 18$
  (C) $4x + 5y = 42$ (D) $4x - 5y = -42$
  (E) $4x - 5y = 42$

Table 1 lists the algebraic skills tested on the 1998–1999 EMPT version of the Mathematical Association of America's Algebra Placement Test and the percent of students answering the items correctly.

The students were classified in one of four levels, on the basis of their total scores, with a suggestion of how their scores would be treated if they had received them on a college-placement test. This analysis was used to advise students on the mathematics courses that they should take in eleventh and twelfth grades. The analysis showed that 18 percent were not ready for college-level courses and should take remedial mathematics; 25 percent had to take remedial mathematics for some majors; 39 percent were ready for a beginning-level college mathematics course but borderline for mathematics, science, and engineering majors; and 18 percent had a solid precollege mathematics preparation.

Table 1 shows a minimal expectation of algebraic skills for entering freshmen in colleges in one state. How students come to acquire and retain these skills, as well as the additional algebra skills that others consider to be basic for postsecondary studies, are not addressed. The development of these skills must be considered as a necessary component in any K–12 algebra program.

Sid Rachlin, rachlins@mail.ecu.edu, teaches mathematics education courses at East Carolina University, where he directs the Middle Math Project. He works with preservice and practicing teachers in middle-grades and secondary mathematics education and previously directed the University of Hawaii's Algebra Learning Project.

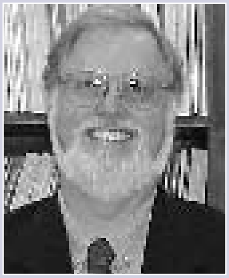

Table 1 — Item Analysis by Decreasing Percent Answering Correctly

Simplify with integral exponents	88%	Simplify using rational exponents	62%	Solve absolute-value inequalities	45%
Simplify using order of operations	83%	Evaluate formula	60%	Graph line giving standard form	44%
Solve linear operation	83%	Solve quadratic equation	60%	Subtract rational expressions	42%
Simplify radical	81%	Solve system of linear equations	60%	Use logarithms to solve for *x*	42%
Simplify using distributive property	78%	Find *y*-intercept	59%	Solve quadratic inequality	40%
Solve linear inequality	70%	Solve fractional equation	56%	Evaluate function	39%
Square binomial	69%	Solve absolute-value equation	55%	Find equation of a line	36%
Rationalize denominator	68%	Solve coin problem	54%	Identify graphs—systems of equations	31%
Simplify rational expression	65%	Solve perimeter problem	52%	Simplify complex fraction	27%
Factor trinomial	63%	Evaluate with negative integral exponents	48%	Evaluate function	25%
Multiply rational expressions	62%	Solve logarithmic equations	46%		

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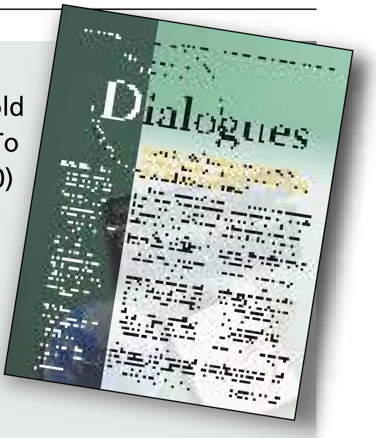

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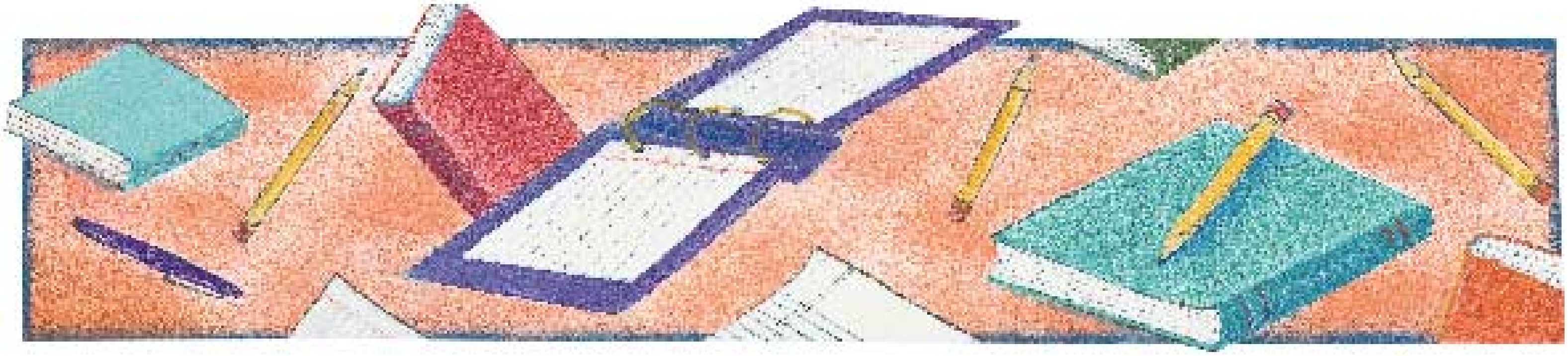


Reader Reactions to the Basics and to Other Issues

by Cynthia Ballheim, *For the Editorial Panel*

Trends emerged in reader responses to questions about basics from the October 1999 issue of *Dialogues*. Almost all responses listed problem solving as the most important of the basics. Eighty-two of the more than 140 respondents thought that basic proficiency in arithmetic was necessary by the end of the elementary grades. Prealgebra skills were necessary by the end of middle school. Calculus was a basic end result for college-intending students; and consumer mathematics, or "survival skills," was the goal for non-college-intending students. Thirteen percent considered all of mathematics basic and unaffected by the use of technology, whereas that same percent believed that logarithms and long division were obsolete. The following representative lists of readers' views indicate overlap between what is basic and what is not.

Basics in the Elementary Grades

- ***Arithmetic:*** Multiplication tables; proficiency at fundamental operations of arithmetic by hand, including decimals and fractions; number sense; estimation; and percents
- ***Algebra:*** Set theory, prealgebra with an introduction to abstract variables, and solving basic equations
- ***Geometry,*** including shapes, relations, and measurement
- ***Statistics***
- ***Other topics listed as basic:*** An emerging problem-solving framework; the courage to go forward in mathematics; thinking puzzles; writing conclusions in sentences and paragraphs; and "tools of the trade," including calculators and shortcuts

Basics at the End of Middle School

- ***Arithmetic:*** Operations on whole numbers, fractions, decimals, and percents; also estimation
- ***Prealgebra and algebra skills,*** including formula manipulation and equations
- ***Geometry,*** with spatial sense and measurement of two- and three-dimensional figures
- ***Probability***
- ***Consumer mathematics***
- ***Critical-thinking skills, proportional reasoning, and problem solving***

Basics for College-Intending High School Graduates

- ***First- and second-year algebra,*** with word problems, graphing calculators, and matrices
- ***Geometry***
- ***Precalculus and calculus***
- ***Iterative and statistical reasoning***
- ***Trigonometry***
- ***Decimals***
- ***Life skills***

Basics for Non-College-Intending Students

- ***Arithmetic:*** Basic arithmetic; number sense; estimation; consumer mathematics, including finance; applied mathematics; and sense of mathematical proportion
- ***Algebra,*** including matrices
- ***Geometry and logical reasoning***
- ***Programming***
- ***Statistical reasoning***
- ***Discrete mathematics***
- ***Probability***
- ***Trigonometry***

Topics No Longer Considered Basic

All math is basic; no nonbasic topics exist.

- ***Arithmetic:*** Multiplying and dividing fractions, and possibly all work with fractions; operations on whole numbers of three digits or more; division by a decimal, and possibly all long division; square roots; prime-factor trees; and paper-and-pencil algorithms in general
- ***Algebra:*** Logarithms, interpolation, roots of polynomials, simplifying radicals, matrices, graphing by hand, factoring, systems of equations, completing the square, and inverses; some felt that algebra was no longer basic.
- ***Geometry:*** Tessellations and geometric proof; some felt that geometry was no longer basic.
- ***Statistics:*** Regression, statistical analysis, pie charts, and bar graphs
- ***Higher mathematics:*** Trigonometry, precalculus, and calculus

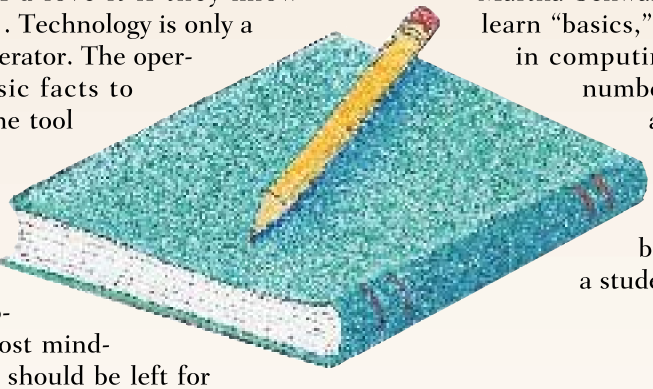
Responses included the following:

Students should have definite life skills concerning writing checks, loans, interest rates on credit cards, budgets, etc. . . . Technology should be used to reinforce skills, not to teach.

Catherine Fernandez
Grades 5–8 Teacher, Illinois

By the end of the sixth grade, I'd love it if they know arithmetic facts "inside and out.". . . Technology is only a tool. A tool is only as good as its operator. The operator must have knowledge of basic facts to know whether "he" has operated the tool correctly.

Rosalind R. Hill
Grades 11–12 Teacher, Idaho



Understanding is more important than content. . . . We must support visual and tactile learners. Most mindless mathematics like computation should be left for technology, leaving meaningful mathematics for students.

Barry M. Hammond
Retired University Teacher, Manitoba

What is really basic for young children is naming quantity, not counting.

Joan A. Cotter
Curriculum Developer, Minnesota

In response to "An Attempt in Sweden at Consensus ":
"On page 15, item 23 [of the May/June issue] . . . , the area of triangle should be $(1/2)ab\sin C$."

Rosalie Triozzi
Grade 8 Teacher, New Jersey

In response to "What's Basic in Math Education: A View from Mathematically Correct":
Martha Schwartz obviously cares that students learn "basics," but her notion that proficiency in computing sums and products of mixed numbers belongs in the same category as starting cars with a hand crank. . . . I would appreciate an explanation of how not knowing the algorithms for mixed-number operations is an indicator that a student will not be good at algebra.

Paul Tisdel
Mathematics Educator, Texas

In response to "Are there topics in mathematics that you are asked to teach that you believe are obsolete?":
As we have sped forward [with the use of technology], I cannot help but ask, "What is being left behind?" I believe that it is a question that we all should investigate; not to bring back the laborious paper-and-pencil calculations and drill, but to not lose the richness behind the processes.

Elizabeth A. Barrett
High School Teacher, New York

(See questionnaire on reverse side of this page)

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