Linear Programming
- A primer

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The feed mix problem

- We want to mix *2* different ingredients in order to prepare a chicken feed.
- Each ingredient *i* has a cost c_i and contains *3* nutrients. The amount of nutrient *j* is a_{ij}
- The total amount of each nutrient *j* in the mix has to be within a lower l_j and upper u_j bound
- How much of each ingredient should we use so that to make *b* amount of mix at minimum cost.

LP formulation

- Let x_1, x_2 be the amount of each ingredient used in the mix.
- Minimize (cost): $c_1x_1+c_2x_2$

  So that:

  $$x_1+x_2 = b$$
  $$l_1 \leq a_{11}x_1+ a_{21}x_2 \leq u_1$$
  $$l_2 \leq a_{12}x_1+ a_{22}x_2 \leq u_1$$
  $$l_3 \leq a_{13}x_1+ a_{23}x_2 \leq u_1$$
  $$x_1, x_2 \geq 0$$

A transportation problem

- Coffee is processed at *2* plants and then shipped to *3* warehouses. Assumptions:
  - Unit shipping cost from plant *i* to warehouse *j* is c_{ij}.
  - Production capacity at plant *i* is a_i
  - Demand at warehouse *j* is b_j.
- How much to produce at each plant and ship to each warehouse so that to minimize transportation cost and meet demands??

LP formulation

- Let x_{ij} be the amount produced at plant *i* and shipped to warehouse *j*. We want to determine the values x_{ij} so that:
- Minimize transportation cost:

$$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}$$

So that:

$$x_{11} + x_{12} + x_{13} \leq a_1$$
$$x_{21} + x_{22} + x_{23} \leq a_2$$

Production constraints

$$x_{11} + x_{21} = b_1$$
$$x_{12} + x_{22} = b_2$$
$$x_{13} + x_{23} = b_3$$

Demand constraints

$$x_{ij} \geq 0$$

The sudoku problem !

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

Solution

9	6	3	1	7	4	2	5	8
1	7	8	3	2	5	6	4	9
2	5	4	6	8	9	7	3	1
8	2	1	4	3	7	5	9	6
4	8	6	8	5	2	3	1	7
7	3	5	9	6	1	8	2	4
5	8	9	7	1	3	4	6	2
3	1	7	2	4	6	9	8	5
6	4	2	5	3	8	1	7	3

Binary integer LP formulation

- Let x_{ijk} be a variable indicating that the (i,j) box contains the value k, k=0,1,..,9
- $x_{ijk} = \begin{cases} 0 \text{ if the } (i,j)^{th} \text{ position does not contain the value } k \\ 1 \text{ if the } (i,j)^{th} \text{ position contains the value } k \end{cases}$
- Objective function

  Max $\sum_i \sum_j \sum_k x_{ijk}$ with i , j = 1,2,…,8, and k = 0,1,…,9

Constraints

- *Row constraints for k value*

  $x_{11k} + x_{12k} + x_{13k} + \ldots + x_{19k} = 1$

  …

  $x_{91k} + x_{92k} + x_{93k} + \ldots + x_{99k} = 1$

- *Column constraints for k value*

  $x_{11k} + x_{12k} + x_{13k} + \ldots + x_{19k} = 1$

  …

  $x_{91k} + x_{92k} + x_{93k} + \ldots + x_{99k} = 1$

- *Box constraints for k value*

  $x_{11k} + x_{12k} + x_{13k} + x_{21k} + x_{22k} + x_{23k} + x_{31k} + x_{32k} + x_{33k} = 1$

  …

  $x_{77k} + x_{78k} + x_{79k} + x_{78k} + x_{88k} + x_{89k} + x_{97k} + x_{98k} + x_{99k} = 1$

LP: A graphical solution

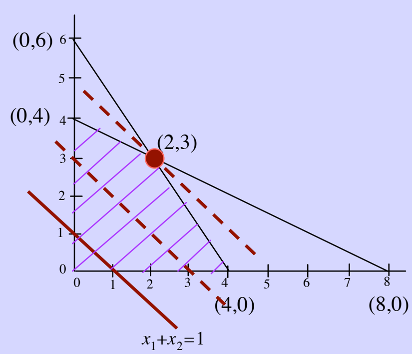
Max P = $x_1 + x_2$

Subject to:

$3x_1 + 2x_2 \leq 12$

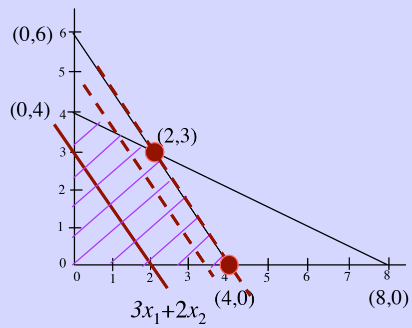
$2x_1 + 4x_2 \leq 16$

$x_1, x_2 \geq 0$



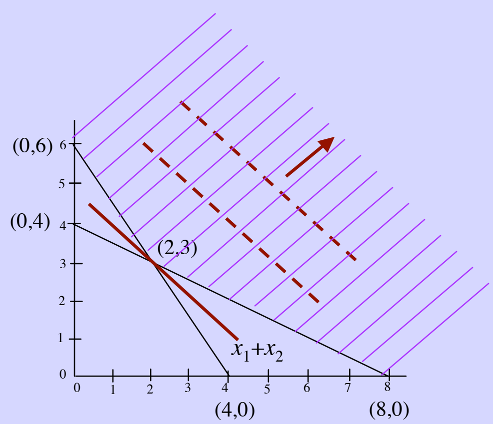
Multiple solutions

Max P = $3x_1+2x_2$
Subject to:
$3x_1 + 2x_2 \leq 12$
$2x_1 + 4x_2 \leq 16$
$x_1, x_2 \geq 0$



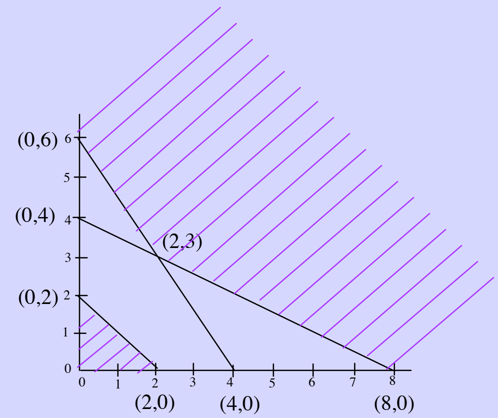
Unbounded solution

Max P = x_1+x_2
Subject to:
$3x_1 + 2x_2 \geq 12$
$2x_1 + 4x_2 \geq 16$
$x_1, x_2 \geq 0$



Empty feasible set

Max P = x_1+x_2
Subject to:
$3x_1 + 2x_2 \geq 12$
$2x_1 + 4x_2 \geq 16$
$x_1+x_2 \leq 2$
$x_1, x_2 \geq 0$



The Simplex algorithm

- The Simplex procedure is an iterative procedure for solving the LP problem.
  - Determines a basic feasible solutions to a system of linear equations
  - Tests the solution for optimality, and then
  - Moves to another basic feasible solution that is better than the previous one.