MA 407: Introduction to Modern Algebra
Homework

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1 Group Theory

1.1 Definition of Group

1. State the definition of group.

2. Is the following a group? If not, why not?

   (a) $(\{0,1,2\}\,,\ +)$

   (b) $(\{0,1,2\}\,,\ \circledcirc)$ where $a \circledcirc b$ is given by

$a\backslash b$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	1

   (c) $(\mathbb{Z}^*,\ +)$ where $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$

   (d) $(\mathbb{Q},\ \times)$

1.2 Examples of Group

1. State the definition of the following notions:

   (a) $\mathbb{Z}_n, +_n$

   (b) $U_n, \times_n$

   (c) $S_n, \circ$

   (d) $D_n, \circ$

2. Construct the operation tables.

   (a) $(\mathbb{Z}_4, \ \ +_4)$

   (b) $(U_8, \ \times_8)$

   (c) $(S_3, \ \ \circ)$

   (d) $(D_4, \ \circ)$

1.3 Uniqueness of Identity and Inverse

1. For each of the following groups, list identities and list inverses for each element

   (a) $(\mathbb{Z}_4, \ \ +_4)$

   (b) $(U_8, \ \times_8)$

   (c) $(S_3, \ \ \circ)$

   (d) $(D_4, \ \circ)$

2. Prove: Let G be a group. Then G has only one identity element.

3. Prove: Let G be a group. Then every element of G has only one inverse.

1.4 Subgroup

1. State the definition of the following notions:

   (a) subgroup

2. For each of the following groups, find all the subgroups.

   (a) $(\mathbb{Z}_4, \ \ +_4)$
   (b) $(U_8, \ \times_8)$
   (c) $(S_3, \ \ \circ)$
   (d) $(D_4, \ \circ)$

3. Prove: Let G be a group and $S \subseteq G$. We have $S \leq G$ iff

   (a) $S \neq \emptyset$
   (b) $\forall a, b \in S \ \ ab^{-1} \in S$.

1.5 Normal subgroup and Quotient group

1. State the definition of the following notions

   (a) normal subgroup

   (b) quotient set

   (c) operation over G/S.

2. For each of the following groups G

   (a) $(\mathbb{Z}_4, \ +_4)$

   (b) $(U_8, \ \times_8)$

   (c) $(S_3, \ \circ)$

   (d) $(D_4, \ \circ)$

   do the followings:

   - Find all the normal subgroups of G.
   - For each normal subgroup S, construct the operation table on G/S.
   - Check if G/S is a group.

3. Prove: Let G be a group and let $S \lhd G$. Then the operation over G/S is well defined, that is, if $aS = a'S$ and $bS = b'S$ then $(ab)\,S = (a'b')\,S$.

4. Prove: Let G be a group and let $S \lhd G$. Then G/S is a group.

1.6 Homomorphism, Isomorphism, Image and Kernel

1. State the definition of the following notions.
   Let $(G, \circ), (G', \circ')$ be groups. Let $\phi : G \longrightarrow G'$.
   (a) Homomorphism
   (b) Isomorphism
   (c) Isomorphic ($\cong$)
   (d) Kernel
   (e) Image

2. For each of the following maps $\phi : (G, \circ) \longrightarrow (G', \circ')$
   (a) $\phi : (\mathbb{Z}_9, +_9) \longrightarrow (\mathbb{Z}_9, +_9)$, given by $x \longmapsto 3 \times_9 x$
   (b) $\phi : (U_8, \times_8) \longrightarrow (U_8, \times_8)$, given by $x \longmapsto \begin{cases} 1 & \text{if } x \text{ is 1 or 3} \\ 5 & \text{otherwise} \end{cases}$
   (c) $\phi : (S_3, \ \circ) \longrightarrow (U_8, \times_8)$, given by $x \longmapsto \begin{cases} 1 & \text{if } x \text{ is an even permutation} \\ 3 & \text{otherwise} \end{cases}$.
   (d) $\phi : (D_4, \ \circ) \longrightarrow (\mathbb{Z}_4, \ +_4)$, given by $x \longmapsto \begin{cases} 0 & \text{if } x \text{ is a rotation} \\ 2 & \text{otherwise} \end{cases}$

   do the followings:

   - Draw the map diagram for ϕ.
   - Verify that ϕ is a homomorphism.
   - Construct the operation table for $\operatorname{im}\phi$, and verify that $\operatorname{im}\phi \ \leq G'$.
   - Construct the operation table for $\ker\phi$, and verify that $\ker\phi \lhd \ G$.
   - Construct the operation table for $G/\ker\phi$.
   - Draw the map diagram for the "natural" isomorphism that shows $G/\ker\phi \cong \operatorname{im}\phi$

3. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $\phi(e) = e'$.

4. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $\forall a \in G \ \ \phi\left(a^{-1}\right) = \phi(a)^{-1'}$.

5. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $\operatorname{im}\phi \ \leq \ G'$.

6. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $\ker\phi \leq \ G$.

7. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $\ker\phi \lhd \ G$.

8. Prove: Let $\phi : (G, \circ) \longrightarrow (G', \circ')$ be a homomorphism. Then $G/\ker\phi \cong \operatorname{im}\phi$.

2 Ring Theory

2.1 Definition of Ring

1. State the definitions of the following abstract notions

   (a) Ring

   (b) Commutative Ring

   (c) Ring with Unity

   (d) Commutative Ring with Unity (CRU)

   (e) Integral domain

   (f) Field

2.2 Examples of Ring

1. State the definitions of the following concrete notations.

   (a) $k\mathbb{Z}$

   (b) $M_n(S)$

   (c) $S[x]$

   (d) $S(x)$

2. Classify the following algebraic structures, using a Venn diagram (as we have done in the class).

$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Z}_3$	$\mathbb{Z}_6$	$3\mathbb{Z}$
$M_2(\mathbb{N})$	$M_2(\mathbb{Z})$	$M_2(\mathbb{Q})$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{Z}_3)$	$M_2(\mathbb{Z}_6)$	$M_2(3\mathbb{Z})$
$\mathbb{N}[x]$	$\mathbb{Z}[x]$	$\mathbb{Q}[x]$	$\mathbb{R}[x]$	$\mathbb{C}[x]$	$\mathbb{Z}_3[x]$	$\mathbb{Z}_6[x]$	$3\mathbb{Z}[x]$
		$\mathbb{Q}(x)$	$\mathbb{R}(x)$	$\mathbb{C}(x)$	$\mathbb{Z}_3(x)$		

2.3 Uniqueness of identity and inverse

1. Prove: Let R be a ring. Then there is only one additive identity.

2. Prove: Let R be a ring. Then every element of R has has only one additive inverse.

3. Prove: Let R be a ring with unity. Then there is only one multiplicative identity.

4. Prove: Let R be a field. Then every non-zero element of R has only one multiplicative inverse.

2.4 Subring

1. State the definitions of the following notions:

   (a) Subring

2. Check the truth of the followings.

   (a) $3\mathbb{Z} \leq \mathbb{Z}$
   (b) $\{0,5\} \ \leq \mathbb{Z}_{12}$
   (c) $2\mathbb{Z}_{12} \leq \mathbb{Z}_{12}$
   (d) $3\mathbb{Z}_{12} \leq \mathbb{Z}_{12}$
   (e) $4\mathbb{Z}_{12} \leq \mathbb{Z}_{12}$
   (f) $6\mathbb{Z}_{12} \leq \mathbb{Z}_{12}$
   (g) $\{a+bi \in \mathbb{C} : a,b \in \mathbb{Z}\} \ \leq \mathbb{C}$
   (h) $\left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \leq M_2\left(\mathbb{R}\right)$

3. Prove: Let R be a ring and $S \subseteq R$. We have $S \leq R$ if

   (a) $S \neq \emptyset$
   (b) $\forall a,b \in S \quad a+(-b) \in S$
   (c) $\forall a,b \in S \quad a \cdot b \in S$

2.5 Ideal and Quotient ring

1. State the definitions of the following notions:
   (a) Ideal
   (b) Generated set
   (c) Quotient set
   (d) Operation on quotient set

2. Check the truth of the followings.
   (a) $3\mathbb{Z} \lhd \mathbb{Z}$
   (b) $\{0,5\} \lhd \mathbb{Z}_{12}$
   (c) $2\mathbb{Z}_{12} \lhd \mathbb{Z}_{12}$
   (d) $3\mathbb{Z}_{12} \lhd \mathbb{Z}_{12}$
   (e) $4\mathbb{Z}_{12} \lhd \mathbb{Z}_{12}$
   (f) $6\mathbb{Z}_{12} \lhd \mathbb{Z}_{12}$
   (g) $\mathbb{Z}[i] = \{a+bi : a,b \in \mathbb{Z}\} \lhd \mathbb{C}$
   (h) $\left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \lhd M_2(\mathbb{R})$

3. List the elements of the following sets:
   (a) $\langle 3 \rangle$ as an ideal of $\mathbb{Z}$
   (b) $\langle 8, 12 \rangle$ as an ideal of $\mathbb{Z}$
   (c) $\langle 2 \rangle$ as an ideal of $\mathbb{Z}_8$
   (d) $\langle 4 \rangle$ as an ideal of $\mathbb{Z}_8$
   (e) $\langle x \rangle$ as an ideal of $\mathbb{Z}_2[x]$
   (f) $\langle x^2 \rangle$ as an ideal of $\mathbb{Z}_2[x]$

4. For each of the following structures
   (a) $\mathbb{Z}/\langle 3 \rangle$
   (b) $\mathbb{Z}_8/\langle 2 \rangle$
   (c) $\mathbb{Z}_8/\langle 4 \rangle$
   (d) $\mathbb{Z}_2[x]/\langle x \rangle$
   (e) $\mathbb{Z}_2[x]/\langle x^2 \rangle$

   do the followings:

   - List the elements.
   - Construct the operation tables for addition and multiplication
   - Verify that it is a ring.

5. Prove: Let R be a CRU and let $a_1, \ldots, a_n \in R$. Then $\langle a_1, \ldots, a_n \rangle \lhd R$.

6. Prove: Let R be a ring and let $I \lhd R$. Then the addition operation on R/I is well defined.

7. Prove: Let R be a ring and let $I \lhd R$. Then the multiplication operation on R/I is well defined.

8. Prove: Let R be a ring and let $I \lhd R$. Then R/I is a ring.

2.6 Homomorphism, Isomorphism, Image and Kernel

1. State the definition of the following notions

   (a) Homomorphism

   (b) Isomorphism

   (c) Isomorphic ($\cong$)

   (d) Kernel

   (e) Image

2. For each of the following maps $\phi : (R, +, \cdot) \longrightarrow (R', +', \cdot')$

   (a) $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_5$, given by $x \longmapsto x \bmod 5$

   (b) $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_{10}$, given by $x \longmapsto (5x) \bmod 10$

   (c) $\phi : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_{10}$, given by $x \longmapsto (6x) \bmod 10$

   do the followings:

   - Draw the map diagram for ϕ.
   - Construct the operation tables for $\operatorname{im} \phi$.
   - Construct the operation tables for $\ker \phi$.
   - Construct the operation tables of $R/\ker \phi$.
   - Draw the map diagram for the "natural" isomorphism that shows $R/\ker \phi \cong \operatorname{im} \phi$

3. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $\phi(0) = 0'$.

4. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $\phi(-a) = -'\phi(a)$.

5. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $\operatorname{im} \phi \leq R'$.

6. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $\ker \phi \leq R$.

7. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $\ker \phi \lhd R$.

8. Prove: Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be rings. Let $\phi : R \longrightarrow R'$ be a homomorphism. Then $R/\ker \phi \cong \operatorname{im} \phi$.