Test 1

1

Solve A x = b using Gauss-Jordan

$$A = \begin{bmatrix} -3 & 3 & -3 \\ -6 & 3 & -5 \\ -6 & -3 & -4 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

1. Augment

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ -6 & 3 & -5 & 0 \\ -6 & -3 & -4 & -3 \end{bmatrix}$$

2. Forward eliminate

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -9 & 2 & -3 \end{bmatrix}, \begin{bmatrix} \\ R2 - 2\,R1 \\ R3 - 2\,R1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}, \begin{bmatrix} \\ \\ R3 - 3\,R2 \end{bmatrix}$$

3. Backward eliminate

$$\begin{bmatrix} -3 & 3 & 0 & 9 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix}, \begin{bmatrix} R1 - 3\,R3 \\ R2 + R3 \\ \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 6 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix}, \begin{bmatrix} R1 + R2 \\ \\ \end{bmatrix}$$

4. Scale

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5. Solution

$$x = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

2

Solve A x = b using Gauss-Jordan

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & 0 \\ 3 & -1 & 9 \end{bmatrix}, b = \begin{bmatrix} -7 \\ 15 \\ -5 \end{bmatrix}$$

1. Augment

$$\begin{bmatrix} -1 & -1 & 1 & -7 \\ 3 & 2 & 0 & 15 \\ 3 & -1 & 9 & -5 \end{bmatrix}$$

2. Forward eliminate

$$\begin{bmatrix} -1 & -1 & 1 & -7 \\ 0 & -1 & 3 & -6 \\ 0 & -4 & 12 & -26 \end{bmatrix}, \begin{bmatrix} \\ R2 + 3\,R1 \\ R3 + 3\,R1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 & -7 \\ 0 & -1 & 3 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} \\ \\ R3 - 4\,R2 \end{bmatrix}$$

3. Solution: None!

3

Solve A x = b using Gauss-Jordan

$$A = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -14 & -19 \\ -3 & 8 & 13 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 5 \\ -5 \end{bmatrix}$$

1. Augment

$$\begin{bmatrix} -3 & 4 & 5 & -1 \\ 9 & -14 & -19 & 5 \\ -3 & 8 & 13 & -5 \end{bmatrix}$$

2. Forward eliminate

$$\begin{bmatrix} -3 & 4 & 5 & -1 \\ 0 & -2 & -4 & 2 \\ 0 & 4 & 8 & -4 \end{bmatrix}, \begin{bmatrix} \\ R2 + 3\,R1 \\ R3 - R1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 & 5 & -1 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \\ \\ R3 + 2\,R2 \end{bmatrix}$$

3. Backward eliminate

$$\begin{bmatrix} -3 & 0 & -3 & 3 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} R1 + 2\,R2 \\ \\ \end{bmatrix}$$

4. Scale

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Solution

$$x = t_3 \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

4

Compute the determinant of A using the definition

$$A = \begin{bmatrix} 5 & 2 \\ 6 & -2 \end{bmatrix}$$

0. Prepare

$$det(A) = (5\,e_1 + 6\,e_2)\,(2\,e_1 - 2\,e_2)$$

1. Apply the multilinearity

$$det(A) = 10\,det(e_1, e_1) - 10\,det(e_1, e_2) + 12\,det(e_2, e_1) - 12\,det(e_2, e_2)$$

2. Apply the antisymmetry

$$det(A) = -22\,det(e_1, e_2)$$

3. Apply the normality

$$det(A) = -22$$

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Solve  A x = b  using Cramer
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$$A = \begin{bmatrix} 2 & 2 \\ -8 & -6 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

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1. Denominator
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$$det\left(\begin{bmatrix} 2 & 2 \\ -8 & -6 \end{bmatrix}\right) = 4$$

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2. Numerators
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$$det\left(\begin{bmatrix} 0 & 2 \\ -2 & -6 \end{bmatrix}\right) = 4$$

$$det\left(\begin{bmatrix} 2 & 0 \\ -8 & -2 \end{bmatrix}\right) = -4$$

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3. Solution
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$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

6

Invert A using determinant

$$A = \begin{bmatrix} -3 & 4 & -3 \\ 3 & -5 & 0 \\ -6 & 9 & -4 \end{bmatrix}$$

1. Determinant of A

$$det\left(\begin{bmatrix} -3 & 4 & -3 \\ 3 & -5 & 0 \\ -6 & 9 & -4 \end{bmatrix}\right) = -3$$

2. Adjugate matrix of A

$$\begin{bmatrix} det\left(\begin{bmatrix} -5 & 0 \\ 9 & -4 \end{bmatrix}\right) & -det\left(\begin{bmatrix} 4 & -3 \\ 9 & -4 \end{bmatrix}\right) & det\left(\begin{bmatrix} 4 & -3 \\ -5 & 0 \end{bmatrix}\right) \\ -det\left(\begin{bmatrix} 3 & 0 \\ -6 & -4 \end{bmatrix}\right) & det\left(\begin{bmatrix} -3 & -3 \\ -6 & -4 \end{bmatrix}\right) & -det\left(\begin{bmatrix} -3 & -3 \\ 3 & 0 \end{bmatrix}\right) \\ det\left(\begin{bmatrix} 3 & -5 \\ -6 & 9 \end{bmatrix}\right) & -det\left(\begin{bmatrix} -3 & 4 \\ -6 & 9 \end{bmatrix}\right) & det\left(\begin{bmatrix} -3 & 4 \\ 3 & -5 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} 20 & -11 & -15 \\ 12 & -6 & -9 \\ -3 & 3 & 3 \end{bmatrix}$$

3. Divide

$$inverse(A) = \begin{bmatrix} -\frac{20}{3} & \frac{11}{3} & 5 \\ -4 & 2 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$. Note

$$\begin{aligned} Ax &= b \\ \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= b \\ x_1 a_1 + x_2 a_2 + x_3 a_3 &= b \end{aligned}$$

Note

$$\begin{aligned} \det(x_1 a_1 + x_2 a_2 + x_3 a_3, a_2, a_3) &= \det(b, a_2, a_3) \\ x_1 \det(a_1, a_2, a_3) + x_2 \det(a_2, a_2, a_3) + x_3 \det(a_3, a_2, a_3) &= \det(b, a_2, a_3) \quad \text{(by multilinearity)} \\ x_1 \det(a_1, a_2, a_3) &= \det(b, a_2, a_3) \quad \text{(by antisymmetry)} \\ x_1 &= \frac{\det(b, a_2, a_3)}{\det(a_1, a_2, a_3)} \\ \det(a_1, x_1 a_1 + x_2 a_2 + x_3 a_3, a_3) &= \det(a_1, b, a_3) \\ x_1 \det(a_1, a_1, a_3) + x_2 \det(a_1, a_2, a_3) + x_3 \det(a_1, a_3, a_3) &= \det(a_1, b, a_3) \quad \text{(by multilinearity)} \\ x_2 \det(a_1, a_2, a_3) &= \det(a_1, b, a_3) \quad \text{(by antisymmetry)} \\ x_2 &= \frac{\det(a_1, b, a_3)}{\det(a_1, a_2, a_3)} \\ \det(a_1, a_2, x_1 a_1 + x_2 a_2 + x_3 a_3) &= \det(a_1, a_2, b) \\ x_1 \det(a_1, a_2, a_1) + x_2 \det(a_1, a_2, a_2) + x_3 \det(a_1, a_2, a_3) &= \det(a_1, a_2, b) \quad \text{(by multilinearity)} \\ x_3 \det(a_1, a_2, a_3) &= \det(a_1, a_2, b) \quad \text{(by antisymmetry)} \\ x_3 &= \frac{\det(a_1, a_2, b)}{\det(a_1, a_2, a_3)} \end{aligned}$$

8. Note

$$\begin{aligned} 0 &= f(a+b, a+b) \\ &= f(a, a+b) + f(b, a+b) \\ &= f(a,a) + f(a,b) + f(b,a) + f(b,b) \\ &= 0 + f(a,b) + f(b,a) + 0 \end{aligned}$$

Hence

$$f(a,b) = -f(b,a)$$