MA 341
Test 3
(Nonlinear Ordinary Differential Equations)

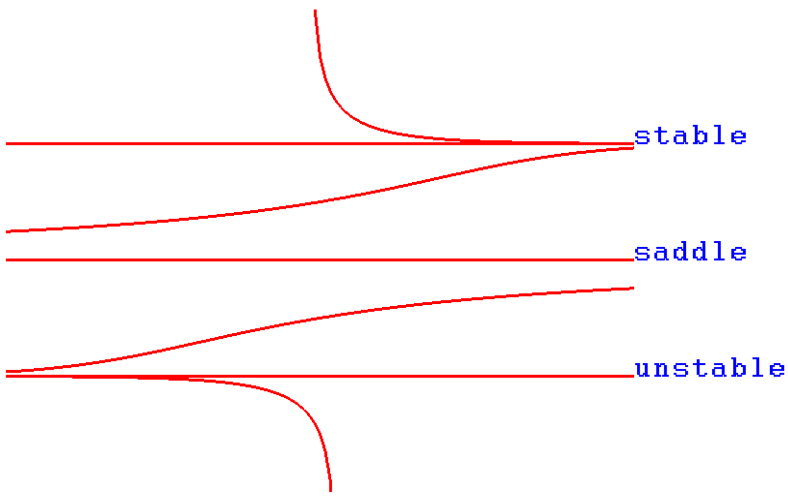
Hoon Hong

1.

Problem:

$$y' = -(y-1)^2 (y+1)(y-3)$$

Solution:

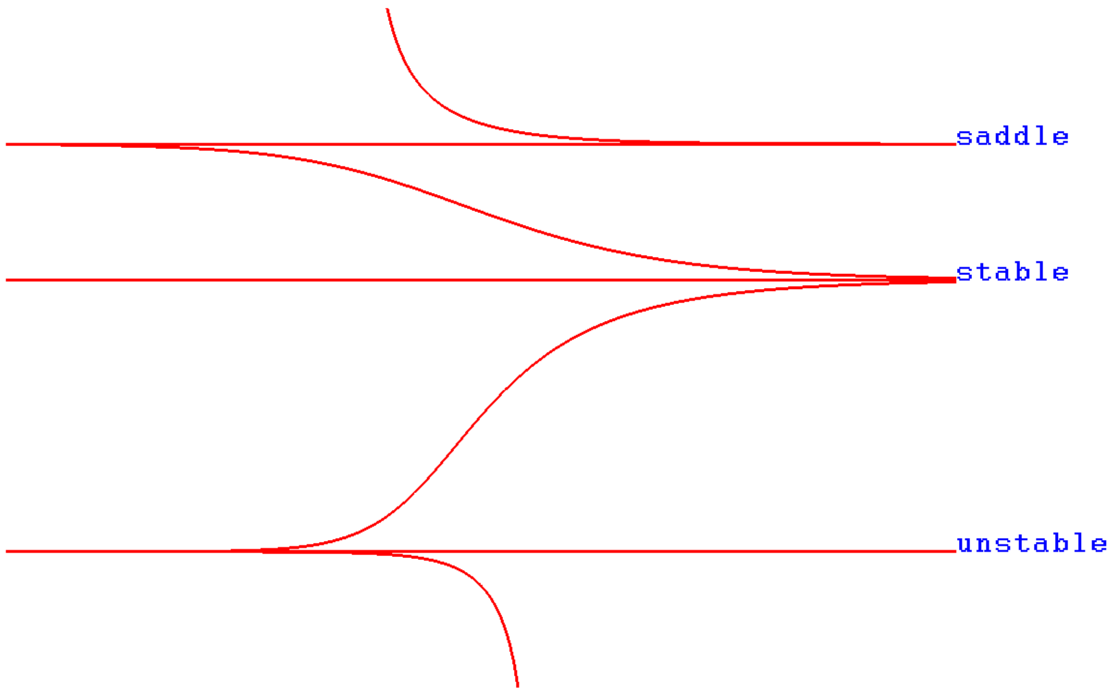


2.

Problem:

$$y' = -|y-4|(y^2-4)$$

Solution:



3.

Problem:

$$y1' = y1^2 - y2$$
$$y2' = y1^2 + y2^2 - 2$$

1. Equilibriums:

$$\begin{bmatrix} -1 & y1^2 & 0 \\ 0 & -1 & y1^2 \\ 1 & 0 & y1^2 - 2 \end{bmatrix}$$

$$y1^2 - 2 + y1^4 = 0$$
$$\{y1 = 1\}, \{y1 = -1\}$$

$$\{y1 = 1\}$$
$$1 - y2 = 0$$
$$-1 + y2^2 = 0$$
$$\{y2 = 1\}$$

$$\{y1 = -1\}$$
$$1 - y2 = 0$$
$$-1 + y2^2 = 0$$
$$\{y2 = 1\}$$

$$[1, 1], [-1, 1]$$

2. Jacobian:

$$J = \begin{bmatrix} 2\,y1 & -1 \\ 2\,y1 & 2\,y2 \end{bmatrix}$$

3. Around [1, 1]:

$$z' = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} z$$
$$\lambda = 2 + \sqrt{2}\, I$$
$$\lambda = 2 - \sqrt{2}\, I$$

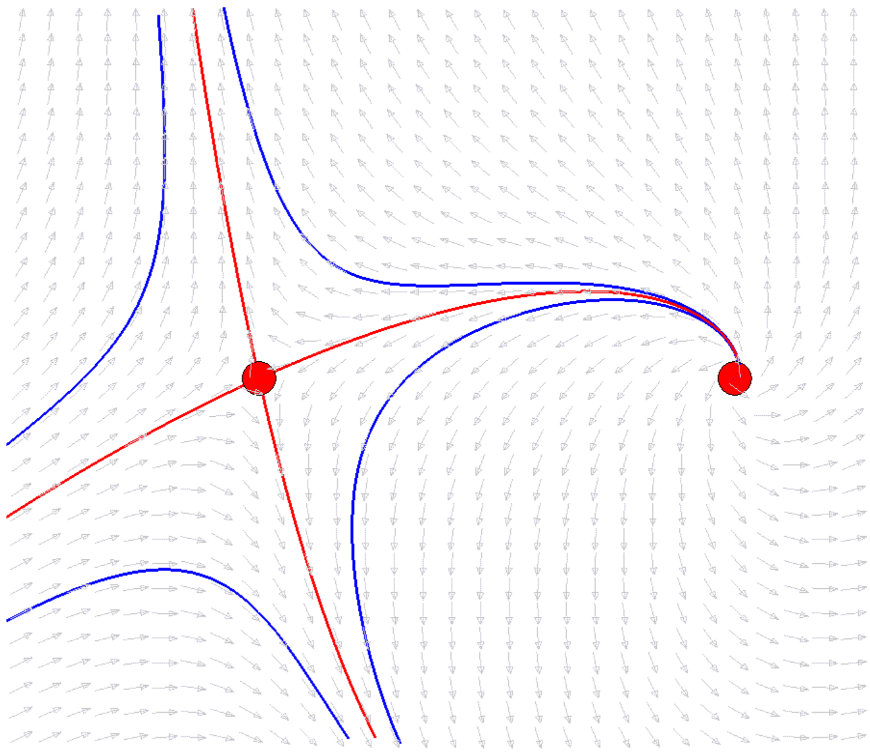
unstable

4. Around [-1, 1]:

$$z' = \begin{bmatrix} -2 & -1 \\ -2 & 2 \end{bmatrix} z$$
$$\lambda = \sqrt{6}, v = [-1, 2 + \sqrt{6}]$$
$$\lambda = -\sqrt{6}, v = [-1, 2 - \sqrt{6}]$$

saddle

5. Sketch:



4.

```
Problem:
```

$$y1' = -y1\,y2 + y2^2 - y1^2 + 4$$
$$y2' = -y1\,y2 - y2^2$$

```
1. Equilibriums:
```

$$\begin{bmatrix} 1 & -y1 & -y1^2+4 & 0 \\ 0 & 1 & -y1 & -y1^2+4 \\ -1 & -y1 & 0 & 0 \\ 0 & -1 & -y1 & 0 \end{bmatrix}$$

$-y1^4 + 16 = 0$

$\{y1 = 2\}, \{y1 = -2\}$

$\{y1 = 2\}$

$-2\,y2 + y2^2 = 0$

$-2\,y2 - y2^2 = 0$

$\{y2 = 0\}$

$\{y1 = -2\}$

$2\,y2 + y2^2 = 0$

$2\,y2 - y2^2 = 0$

$\{y2 = 0\}$

$[2, 0], [-2, 0]$

```
2. Around [2, 0]:
```

$y1 = z1 + 2$

$y2 = z2$

$z1' = -(z1 + 2)\,z2 + z2^2 - (z1 + 2)^2 + 4$

$z2' = -(z1 + 2)\,z2 - z2^2$

$z1' = -z2\,z1 - 2\,z2 + z2^2 - z1^2 - 4\,z1$

$z2' = -z2\,z1 - 2\,z2 - z2^2$

$z1' = -4\,z1 - 2\,z2$

$z2' = -2\,z2$

$$z' = \begin{bmatrix} -4 & -2 \\ 0 & -2 \end{bmatrix} z$$

$\lambda = -4, v = [-2, 0]$

$\lambda = -2, v = [-2, 2]$

stable

```
3. Around [-2, 0]:
```

$y1 = z1 - 2$

$y2 = z2$

$z1' = -(z1-2)\,z2 + z2^2 - (z1-2)^2 + 4$

$z2' = -(z1-2)\,z2 - z2^2$

$z1' = -z2\,z1 + 2\,z2 + z2^2 - z1^2 + 4\,z1$

$z2' = -z2\,z1 + 2\,z2 - z2^2$

$z1' = 4\,z1 + 2\,z2$

$z2' = 2\,z2$

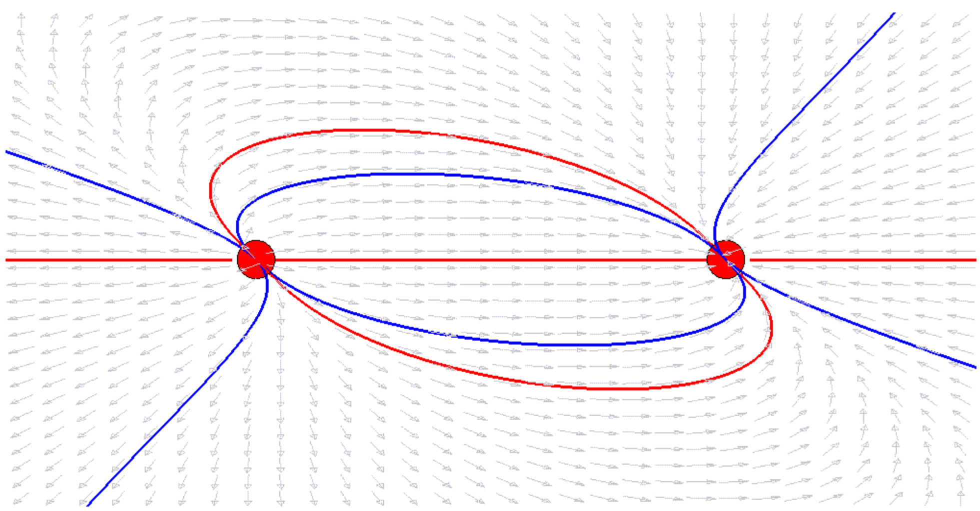
$z' = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} z$

$\lambda = 4, v = [2, 0]$

$\lambda = 2, v = [2, -2]$

unstable

4. Sketch:



5.

Problem:

$$y' = (y-2)(y-5)\sin(t)$$
$$y(0) = 0$$

1. Find the general solution:

$$\frac{dy}{(y-2)(y-5)} = \sin(t)\,dt$$
$$\left(-\frac{1}{3(y-2)} + \frac{1}{3(y-5)}\right) dy = \sin(t)\,dt$$
$$-\frac{1}{3}\ln(|y-2|) + \frac{1}{3}\ln(|y-5|) = -\cos(t) + C$$
$$-\ln(|y-2|) + \ln(|y-5|) = -3\cos(t) + C$$
$$\ln\left(\frac{|y-5|}{|y-2|}\right) = -3\cos(t) + C$$
$$\frac{|y-5|}{|y-2|} = \mathbf{e}^{(-3\cos(t)+C)}$$
$$\frac{|y-5|}{|y-2|} = C\,\mathbf{e}^{(-3\cos(t))},\ \ C > 0$$

2. Find the particular solution:

$$C = \frac{5}{2}\mathbf{e}^{3}$$
$$\frac{|y-5|}{|y-2|} = \frac{5}{2}\mathbf{e}^{3}\,\mathbf{e}^{(-3\cos(t))}$$

6.

Problem:

$$y1' = -\frac{e^{y1}\cos(y2)}{(y2-1)(y2-2)}$$

$$y2' = \frac{(y1+1)e^{y1}\cos(y2)}{(y1-3)(y1-5)}$$

$$y1(0) = 4$$

$$y2(0) = 0$$

1. Find the general solution:

$$dy1/dy2 = -\frac{(y1-3)(y1-5)}{(y2-1)(y2-2)(y1+1)}$$

$$\frac{(y1+1)\,dy1}{(y1-3)(y1-5)} = -\frac{dy2}{(y2-1)(y2-2)}$$

$$\left(\frac{3}{y1-5} - \frac{2}{y1-3}\right)dy1 = \left(\frac{1}{y2-1} - \frac{1}{y2-2}\right)dy2$$

$$3\ln(|y1-5|) - 2\ln(|y1-3|) = \ln(|y2-1|) - \ln(|y2-2|) + C$$

$$\ln\left(\frac{|y1-5|^3}{|y1-3|^2}\right) = \ln\left(\frac{|y2-1|}{|y2-2|}\right) + C$$

$$\frac{|y1-5|^3}{|y1-3|^2} = \frac{C|y2-1|}{|y2-2|},\ C > 0$$

2. Find the particular solution:

$$1 = \frac{C}{2}$$

$$C = 2$$

$$\frac{|y1-5|^3}{|y1-3|^2} = \frac{2|y2-1|}{|y2-2|}$$