MA 341
Test 2
(Non-homogeneous Linear Ordinary Differential Equations)

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1. Undetermined Coefficient Method: 2nd-order, 1 variable

Problem:

$$y'' - 7y' + 12y = -36$$

$$y(0) = 0,\ y'(0) = 0$$

1. Find the general solution of the homogeneous equation:

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\lambda_1 = 4,\ \lambda_2 = 3$$

$$y_g = C_1 \mathbf{e}^{(4t)} + C_2 \mathbf{e}^{(3t)}$$

2. Find a particular soultion of the non-homogeneous equation:

$$y = \mathrm{D}$$

$$12\,\mathrm{D} = -36$$

$$\mathrm{D} = -3$$

$$y_p = -3$$

3. Find the general solution of the non-homogeneous equation:

$$y = C_1 \mathbf{e}^{(4t)} + C_2 \mathbf{e}^{(3t)} - 3$$

4. Find the particular solution of the non-homogeneous equation:

$$y' = 4C_1 \mathbf{e}^{(4t)} + 3C_2 \mathbf{e}^{(3t)}$$

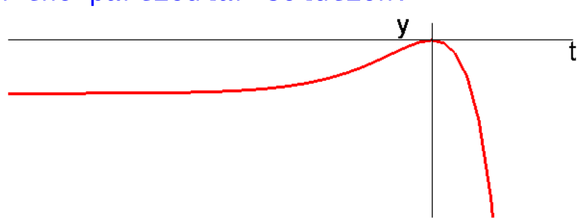
$$0 = C_1 + C_2 - 3$$

$$0 = 4C_1 + 3C_2$$

$$C_1 = -9,\ C_2 = 12$$

$$y = -9\,\mathbf{e}^{(4t)} + 12\,\mathbf{e}^{(3t)} - 3$$

5. Sketch the particular solution:



2. Laplace Transform Method: 2nd-order, 1 variable, Exponential function, Real eigenvalues

Problem:

$$y'' + 9\, y' + 20\, y = -36\, \mathbf{e}^{(-t)}$$

$$y(0) = 0,\ y'(0) = 0$$

1. Laplace Transform:

$$s^2\, Y + 9\, s\, Y + 20\, Y = -\frac{36}{s+1}$$

2. Solve for Y:

$$Y = -\frac{36}{(s^2 + 9\, s + 20)\,(s+1)}$$

3. Inverse Laplace Transform:

$$-\frac{36}{(s+4)\,(s+5)\,(s+1)} = \frac{A}{s+4} + \frac{B}{s+5} + \frac{C}{s+1}$$

$$-36 = A\,(s+5)\,(s+1) + B\,(s+4)\,(s+1) + C\,(s+4)\,(s+5)$$

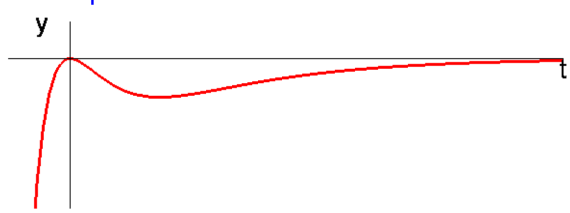
$$s = -4,\ -36 = -3\, A,\ A = 12$$

$$s = -5,\ -36 = 4\, B,\ B = -9$$

$$s = -1,\ -36 = 12\, C,\ C = -3$$

$$y = 12\, \mathbf{e}^{(-4\, t)} - 9\, \mathbf{e}^{(-5\, t)} - 3\, \mathbf{e}^{(-t)}$$

4. Sketch the particular solution:



3. Laplace Transform Method: 2nd-order, 1 variable, Exponential function, Non-real eigenvalues

Problem:

$$y'' - 2y' + 10y = -60$$

$$y(0) = 0,\ y'(0) = 0$$

1. Laplace Transform:

$$s^2 Y - 2 s Y + 10 Y = -\frac{60}{s}$$

2. Solve for Y:

$$Y = -\frac{60}{(s^2 - 2s + 10)\, s}$$

3. Inverse Laplace Transform:

$$\alpha = 1,\ \beta = 3$$

$$-\frac{60}{((s-1)^2 + 9)\, s} = \frac{A(s-1) + 3B}{(s-1)^2 + 9} + \frac{C}{s}$$

$$-60 = (A(s-1) + 3B)\, s + C((s-1)^2 + 9)$$

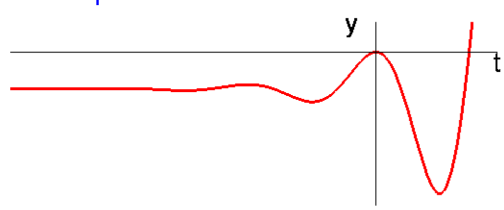
$$s = 0,\ -60 = 10C,\ C = -6$$

$$s = 1,\ -60 = 3B + 9C,\ B = -2$$

$$s = 2,\ -60 = 2A + 6B + 10C,\ A = 6$$

$$y = \mathbf{e}^t (6\cos(3t) - 2\sin(3t)) - 6$$

4. Sketch the particular solution:



4. Laplace Transform Method: 1st-order, 2 variables, Exponential function

Problem:

$$y' = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} y + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \mathbf{e}^{(2t)}$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} Y + \frac{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}{-2+s}$$

2. Solve for Y:

$$\begin{bmatrix} -2+s & 2 \\ 1 & -3+s \end{bmatrix} Y = \frac{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}{-2+s}$$

$$Y = \frac{\begin{bmatrix} -16+4s \\ 2s-8 \end{bmatrix}}{(4-5s+s^2)(-2+s)}$$

3. Inverse Laplace Transform:

$$\frac{\begin{bmatrix} -16+4s \\ 2s-8 \end{bmatrix}}{(s-1)(-4+s)(-2+s)} = \frac{A}{-4+s} + \frac{B}{s-1} + \frac{C}{-2+s}$$

$$\begin{bmatrix} -16+4s \\ 2s-8 \end{bmatrix} = A(s-1)(-2+s) + B(-4+s)(-2+s) + C(-4+s)(s-1)$$

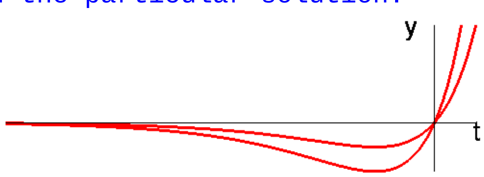
$$s = 4, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 6A,\ A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$s = 1, \begin{bmatrix} -12 \\ -6 \end{bmatrix} = 3B,\ B = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$s = 2, \begin{bmatrix} -8 \\ -4 \end{bmatrix} = -2C,\ C = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{e}^{(4t)} + \begin{bmatrix} -4 \\ -2 \end{bmatrix} \mathbf{e}^{t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \mathbf{e}^{(2t)}$$

4. Sketch the particular solution:



5. Laplace Transform Method: 1st-order, 1 variable, Trigonometric function

Problem:

$$y' - y = -26 \sin(5\,t)$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y - Y = -\frac{130}{s^2 + 25}$$

2. Solve for Y:

$$Y = -\frac{130}{(s-1)\,(s^2+25)}$$

3. Inverse Laplace Transform:

$$-\frac{130}{(s-1)\,(s^2+25)} = \frac{A}{s-1} + \frac{B\,s + 5\,C}{s^2+25}$$

$$-130 = A\,(s^2+25) + (B\,s + 5\,C)\,(s-1)$$

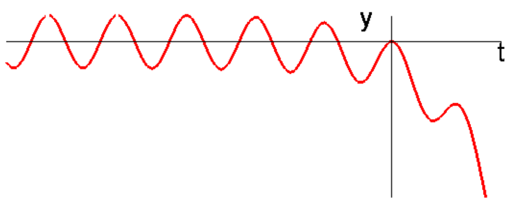
$$s = 1,\ -130 = 26\,A,\ A = -5$$

$$s = 0,\ -130 = 25\,A - 5\,C,\ C = 1$$

$$s = 2,\ -130 = 29\,A + 2\,B + 5\,C,\ B = 5$$

$$y = -5\,\mathbf{e}^{t} + 5\cos(5\,t) + \sin(5\,t)$$

4. Sketch the particular solution:



6. Laplace Transform Method: 1st-order, 1 variable, Step function

Problem:

$$y' + 3\,y = -6\,\mathrm{u}(t-2)$$

$$\mathrm{y}(0) = 0$$

1. Laplace Transform:

$$s\,Y + 3\,Y = -\frac{6\,\mathbf{e}^{(-2\,s)}}{s}$$

2. Solve for Y:

$$Y = -\frac{6\,\mathbf{e}^{(-2\,s)}}{(s+3)\,s}$$

3. Inverse Laplace Transform:

$$-\frac{6}{(s+3)\,s} = \frac{A}{s+3} + \frac{B}{s}$$

$$-6 = A\,s + B\,(s+3)$$

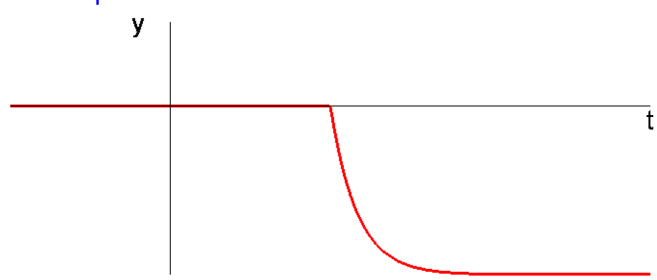
$$s = -3,\ -6 = -3\,A,\ A = 2$$

$$s = 0,\ -6 = 3\,B,\ B = -2$$

$$f = 2\,\mathbf{e}^{(-3\,t)} - 2$$

$$y = (2\,\mathbf{e}^{(-3\,t+6)} - 2)\,\mathrm{u}(t-2)$$

4. Sketch the particular solution:



7. Laplace Transform Method: 1st-order, 1 variable, Impulse function

Problem:

$$y' + 3\,y = -5\,\delta(t-2)$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y + 3\,Y = -5\,\mathbf{e}^{(-2\,s)}$$

2. Solve for Y:

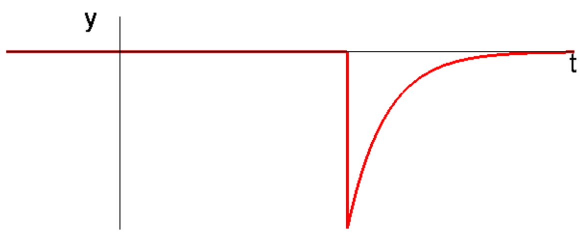
$$Y = -\frac{5\,\mathbf{e}^{(-2\,s)}}{s+3}$$

3. Inverse Laplace Transform:

$$f = -5\,\mathbf{e}^{(-3\,t)}$$

$$y = -5\,\mathbf{e}^{(-3\,t+6)}\,\mathrm{u}(t-2)$$

4. Sketch the particular solution:



8.

Note

$$\begin{aligned}\mathcal{L}\left(e^{(\alpha+i\beta)t} \right) &= \mathcal{L}\left(e^{\alpha t}(\cos\beta t + i\sin\beta t) \right) \\ &= \mathcal{L}\left(e^{\alpha t}\cos\beta t \right) + i\mathcal{L}\left(e^{\alpha t}\sin\beta t \right)\end{aligned}$$

Note

$$\begin{aligned}\mathcal{L}\left(e^{(\alpha+i\beta)t} \right) &= \frac{1}{s-(\alpha+i\beta)} \\ &= \frac{1}{(s-\alpha)-i\beta} \\ &= \frac{(s-\alpha)+i\beta}{((s-\alpha)-i\beta)((s-\alpha)+i\beta)} \\ &= \frac{(s-\alpha)+i\beta}{(s-\alpha)^2-(i\beta)^2} \\ &= \frac{(s-\alpha)+i\beta}{(s-\alpha)^2+\beta^2} \\ &= \frac{(s-\alpha)}{(s-\alpha)^2+\beta^2} + i\frac{\beta}{(s-\alpha)^2+\beta^2}\end{aligned}$$

Thus

$$\mathcal{L}\left(e^{\alpha t}\cos\beta t \right) = \frac{(s-\alpha)}{(s-\alpha)^2+\beta^2}$$

$$\mathcal{L}\left(e^{\alpha t}\sin\beta t \right) = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

$$\mathcal{L}\left(e^{\alpha t}(A\cos\beta t + B\sin\beta t) \right) = \frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}$$