MA 341
Test 2
(Non-homogeneous Linear Ordinary Differential Equations)

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1. Undetermined Coefficient Method: 2nd-order, 1 variable

Problem:

$$y'' + 7\,y' + 12\,y = 24$$

$$y(0) = 0,\ y'(0) = 0$$

1. Find the general solution of the homogeneous equation:

$$\lambda^2 + 7\,\lambda + 12 = 0$$

$$\lambda_1 = -3,\ \lambda_2 = -4$$

$$y_g = C_1\,\mathbf{e}^{(-3\,t)} + C_2\,\mathbf{e}^{(-4\,t)}$$

2. Find a particular soultion of the non-homogeneous equation:

$$y = \mathrm{D}$$

$$12\,\mathrm{D} = 24$$

$$\mathrm{D} = 2$$

$$y_p = 2$$

3. Find the general solution of the non-homogeneous equation:

$$y = C_1\,\mathbf{e}^{(-3\,t)} + C_2\,\mathbf{e}^{(-4\,t)} + 2$$

4. Find the particular solution of the non-homogeneous equation:

$$y' = -3\,C_1\,\mathbf{e}^{(-3\,t)} - 4\,C_2\,\mathbf{e}^{(-4\,t)}$$

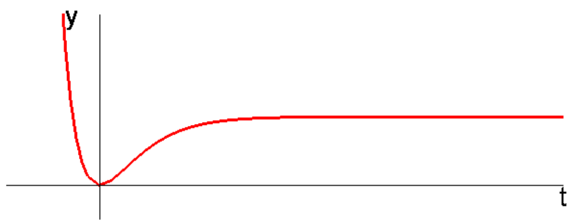
$$0 = C_1 + C_2 + 2$$

$$0 = -3\,C_1 - 4\,C_2$$

$$C_1 = -8,\ C_2 = 6$$

$$y = -8\,\mathbf{e}^{(-3\,t)} + 6\,\mathbf{e}^{(-4\,t)} + 2$$

5. Sketch the particular solution:



2. Laplace Transform Method: 2nd-order, 1 variable, Exponential function, Real eigenvalues

Problem:

$$y'' + 9\,y' + 20\,y = 18\,\mathbf{e}^{(-2\,t)}$$

$$y(0) = 0,\ y'(0) = 0$$

1. Laplace Transform:

$$s^2\,Y + 9\,s\,Y + 20\,Y = \frac{18}{s+2}$$

2. Solve for Y:

$$Y = \frac{18}{(s^2 + 9\,s + 20)\,(s+2)}$$

3. Inverse Laplace Transform:

$$\frac{18}{(s+4)\,(s+5)\,(s+2)} = \frac{A}{s+4} + \frac{B}{s+5} + \frac{C}{s+2}$$

$$18 = A\,(s+5)\,(s+2) + B\,(s+4)\,(s+2) + C\,(s+4)\,(s+5)$$

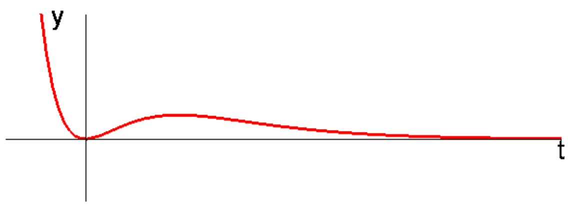
$$s = \text{-}4,\ 18 = -2\,A,\ A = \text{-}9$$

$$s = \text{-}5,\ 18 = 3\,B,\ B = 6$$

$$s = \text{-}2,\ 18 = 6\,C,\ C = 3$$

$$y = -9\,\mathbf{e}^{(-4\,t)} + 6\,\mathbf{e}^{(-5\,t)} + 3\,\mathbf{e}^{(-2\,t)}$$

4. Sketch the particular solution:



3. Laplace Transform Method: 2nd-order, 1 variable, Exponential function, Non-real eigenvalues

Problem:

$$y'' + 2\,y' + 10\,y = -30$$

$$y(0) = 0,\ y'(0) = 0$$

1. Laplace Transform:

$$s^2\,Y + 2\,s\,Y + 10\,Y = -\frac{30}{s}$$

2. Solve for Y:

$$Y = -\frac{30}{(s^2 + 2\,s + 10)\,s}$$

3. Inverse Laplace Transform:

$$\alpha = -1,\ \beta = 3$$

$$-\frac{30}{((s+1)^2 + 9)\,s} = \frac{A\,(s+1) + 3\,B}{(s+1)^2 + 9} + \frac{C}{s}$$

$$-30 = (A\,(s+1) + 3\,B)\,s + C\,((s+1)^2 + 9)$$

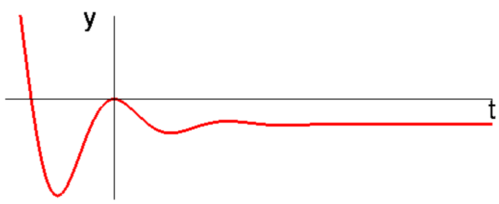
$$s = 0,\ -30 = 10\,C,\ C = -3$$

$$s = -1,\ -30 = -3\,B + 9\,C,\ B = 1$$

$$s = 1,\ -30 = 2\,A + 3\,B + 13\,C,\ A = 3$$

$$y = \mathbf{e}^{(-t)}\,(3\cos(3\,t) + \sin(3\,t)) - 3$$

4. Sketch the particular solution:



4. Laplace Transform Method: 1st-order, 2 variables, Exponential function

Problem:

$$y' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} y + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \mathbf{e}^{(-2\,t)}$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} Y + \frac{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}{s+2}$$

2. Solve for Y:

$$\begin{bmatrix} 3+s & -1 \\ -2 & s+2 \end{bmatrix} Y = \frac{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}{s+2}$$

$$Y = \frac{\begin{bmatrix} -s-4 \\ -8-2\,s \end{bmatrix}}{(5\,s+4+s^2)\,(s+2)}$$

3. Inverse Laplace Transform:

$$\frac{\begin{bmatrix} -s-4 \\ -8-2\,s \end{bmatrix}}{(s+4)\,(s+1)\,(s+2)} = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+2}$$

$$\begin{bmatrix} -s-4 \\ -8-2\,s \end{bmatrix} = A\,(s+4)\,(s+2) + B\,(s+1)\,(s+2) + C\,(s+1)\,(s+4)$$

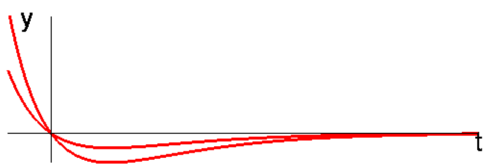
$$s = -1, \begin{bmatrix} -3 \\ -6 \end{bmatrix} = 3\,A,\ A = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$s = -4, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 6\,B,\ B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$s = -2, \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -2\,C,\ C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \mathbf{e}^{(-t)} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{e}^{(-4\,t)} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{e}^{(-2\,t)}$$

4. Sketch the particular solution:



5. Laplace Transform Method: 1st-order, 1 variable, Trigonometric function

Problem:

$$y' + y = -10 \sin(3\,t)$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y + Y = -\frac{30}{s^2 + 9}$$

2. Solve for Y:

$$Y = -\frac{30}{(s+1)\,(s^2+9)}$$

3. Inverse Laplace Transform:

$$-\frac{30}{(s+1)\,(s^2+9)} = \frac{A}{s+1} + \frac{B\,s + 3\,C}{s^2+9}$$

$$\text{-30} = A\,(s^2+9) + (B\,s + 3\,C)\,(s+1)$$

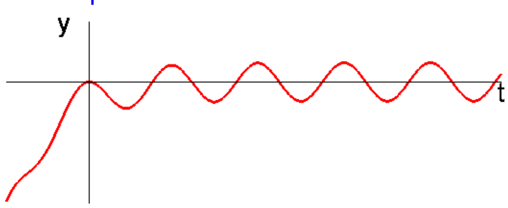
$$s = \text{-1},\ \text{-30} = 10\,A,\ A = \text{-3}$$

$$s = 0,\ \text{-30} = 9\,A + 3\,C,\ C = \text{-1}$$

$$s = 1,\ \text{-30} = 10\,A + 2\,B + 6\,C,\ B = 3$$

$$y = -3\,\mathbf{e}^{(-t)} + 3\cos(3\,t) - \sin(3\,t)$$

4. Sketch the particular solution:



6. Laplace Transform Method: 1st-order, 1 variable, Step function

Problem:

$$y' + y = 3\,\mathbf{e}^{(-2t+2)}\,\mathrm{u}(t-1)$$

$$\mathrm{y}(0) = 0$$

1. Laplace Transform:

$$s\,Y + Y = \frac{3\,\mathbf{e}^{(-s)}}{s+2}$$

2. Solve for Y:

$$Y = \frac{3\,\mathbf{e}^{(-s)}}{(s+1)\,(s+2)}$$

3. Inverse Laplace Transform:

$$\frac{3}{(s+1)\,(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$3 = A\,(s+2) + B\,(s+1)$$

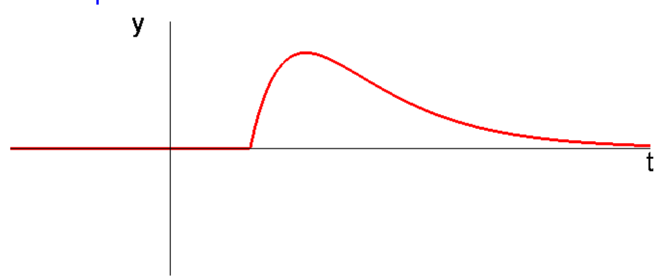
$$s = \text{-}1,\ 3 = A,\ A = 3$$

$$s = \text{-}2,\ 3 = -B,\ B = \text{-}3$$

$$f = 3\,\mathbf{e}^{(-t)} - 3\,\mathbf{e}^{(-2t)}$$

$$y = (3\,\mathbf{e}^{(-t+1)} - 3\,\mathbf{e}^{(-2t+2)})\,\mathrm{u}(t-1)$$

4. Sketch the particular solution:



7. Laplace Transform Method: 1st-order, 1 variable, Impulse function

Problem:

$$y' + 3y = 10\,\delta(t-2)$$

$$y(0) = 0$$

1. Laplace Transform:

$$s\,Y + 3\,Y = 10\,\mathbf{e}^{(-2\,s)}$$

2. Solve for Y:

$$Y = \frac{10\,\mathbf{e}^{(-2\,s)}}{3+s}$$

3. Inverse Laplace Transform:

$$f = 10\,\mathbf{e}^{(-3\,t)}$$

$$y = 10\,\mathbf{e}^{(-3\,t+6)}\,\mathrm{u}(t-2)$$

4. Sketch the particular solution: