MA 341
Test 1
(Homogeneous Linear Ordinary Differential Equations)

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1st-order, 1 variable

Problem:

$y' - 3y = 0$

$y(0) = -6$

1. Find the general solution:

$\lambda - 3 = 0$

$\lambda = 3$

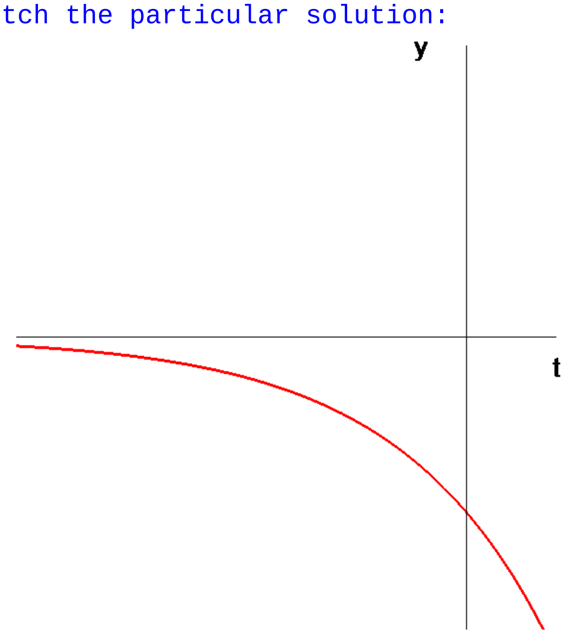
$y = C\,\mathbf{e}^{(3t)}$

2. Find the particular solution:

$C = -6$

$y = -6\,\mathbf{e}^{(3t)}$

3. Sketch the particular solution:



2nd-order, 1 variable: Real eigenvalues

Problem:

$$y'' + 6y' + 8y = 0$$

$$y(0) = 1$$

$$y'(0) = 0$$

1. Find the general solution:

$$\lambda^2 + 6\lambda + 8 = 0$$

$$\lambda = -2, -4$$

$$y = C_1 \mathbf{e}^{(-2t)} + C_2 \mathbf{e}^{(-4t)}$$

2. Find the particular solution:

$$y' = -2C_1 \mathbf{e}^{(-2t)} - 4C_2 \mathbf{e}^{(-4t)}$$

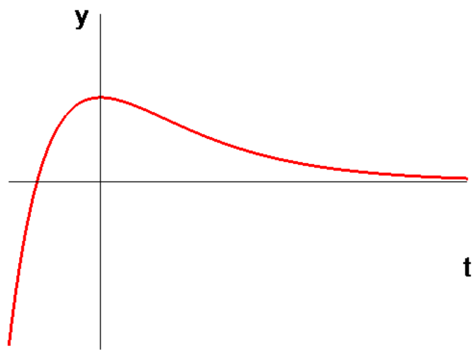
$$C_1 + C_2 = 1$$

$$-2C_1 - 4C_2 = 0$$

$$C_1 = 2, C_2 = -1$$

$$y = 2\mathbf{e}^{(-2t)} - \mathbf{e}^{(-4t)}$$

3. Sketch the particular solution:



2nd-order, 1 variable: Non-real eigenvalues

Problem:

$$y'' - 2y' + 26y = 0$$

$$y(0) = -1$$

$$y'(0) = 4$$

1. Find the general solution:

$$\lambda^2 - 2\lambda + 26 = 0$$

$$\lambda = 1 + 5I, 1 - 5I$$

$$y = \mathbf{e}^t (C_1 \cos(5t) + C_2 \sin(5t))$$

2. Find the particular solution:

$$y' = \mathbf{e}^t (C_1 \cos(5t) + C_2 \sin(5t)) + \mathbf{e}^t (-5 C_1 \sin(5t) + 5 C_2 \cos(5t))$$

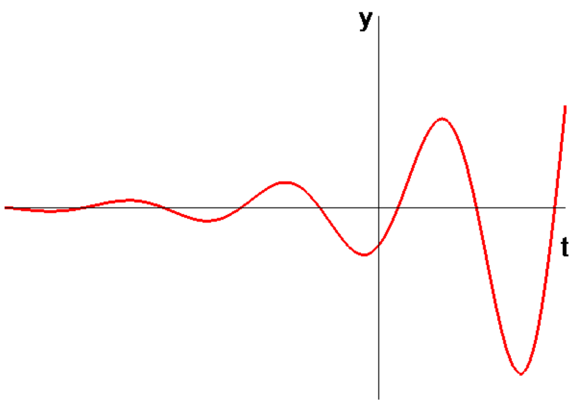
$$C_1 = -1$$

$$C_1 + 5 C_2 = 4$$

$$C_2 = 1$$

$$y = \mathbf{e}^t (-\cos(5t) + \sin(5t))$$

3. Sketch the particular solution:



1st-order, 2 variables: Real eigenvalues

Problem:

$$y' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

1. Find the general solution:

$$Det \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

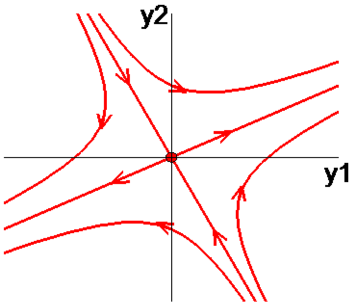
$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda = 3, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -2, v = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$y = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathbf{e}^{(3t)} + C_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} \mathbf{e}^{(-2t)}$$

2. Sketch the general solution:



Saddle

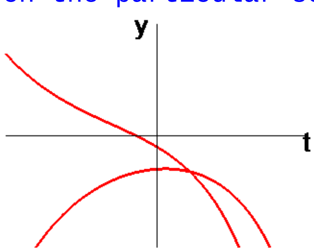
3. Find the particular solution:

$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$C_1 = -2, C_2 = 1$$

$$y = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \mathbf{e}^{(3t)} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} \mathbf{e}^{(-2t)}$$

4. Sketch the particular solution:



Problem:

$$y' = \begin{bmatrix} -3 & -2 \\ -1 & -2 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

1. Find the general solution:

$$Det \begin{bmatrix} -3-\lambda & -2 \\ -1 & -2-\lambda \end{bmatrix} = 0$$

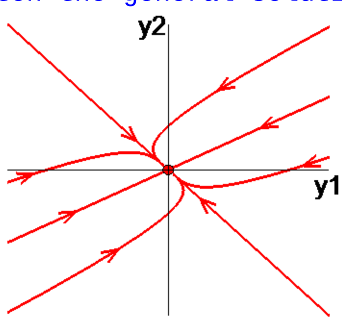
$$\lambda^2 + 5\lambda + 4 = 0$$

$$\lambda = -4, v = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\lambda = -1, v = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$y = C_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} \mathbf{e}^{(-4t)} + C_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \mathbf{e}^{(-t)}$$

2. Sketch the general solution:



Stable

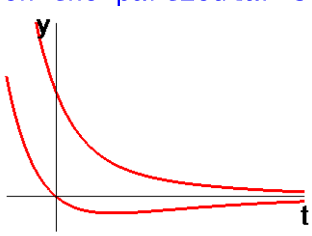
3. Find the particular solution:

$$C_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$C_1 = -2, C_2 = -1$$

$$y = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \mathbf{e}^{(-4t)} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \mathbf{e}^{(-t)}$$

4. Sketch the particular solution:



1st-order 2 variables: Non-real eigenvalues

Problem:

$$y' = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

1. Find the general solution:

$$Det \begin{bmatrix} -1-\lambda & -4 \\ 4 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + 2\lambda + 17 = 0$$

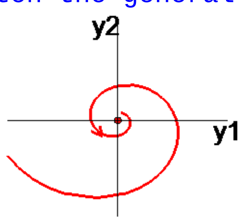
$$\lambda = -1 + 4I,\ v = \begin{bmatrix} -4 \\ 4I \end{bmatrix}$$

$$\lambda = -1 - 4I,\ v = \begin{bmatrix} -4 \\ -4I \end{bmatrix}$$

$$\alpha = -1,\ \beta = 4,\ p = \begin{bmatrix} -4 \\ 0 \end{bmatrix},\ q = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$y = \mathbf{e}^{(-t)} \left(\left(C_1 \begin{bmatrix} -4 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) \cos(4t) + \left(C_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix} - C_1 \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) \sin(4t) \right)$$

2. Sketch the general solution:



Stable

3. Find the particular solution:

$$C_1 \begin{bmatrix} -4 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

$$C_1 = -2,\ C_2 = -1$$

$$y = \mathbf{e}^{(-t)} \left(\begin{bmatrix} 8 \\ -4 \end{bmatrix} \cos(4t) + \begin{bmatrix} 4 \\ 8 \end{bmatrix} \sin(4t) \right)$$

4. Sketch the particular solution: