MA 341
Test 1
(Homogeneous Linear Ordinary Differential Equations)

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1st-order, 1 variable

Problem:

$$y' - 2y = 0$$

$$y(0) = -6$$

1. Find the general solution:

$$\lambda - 2 = 0$$

$$\lambda = 2$$

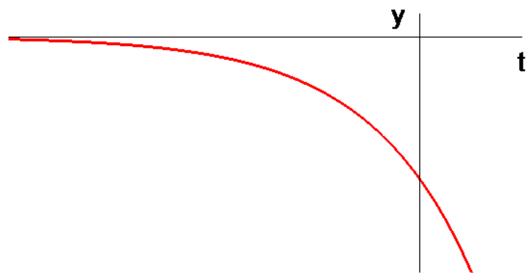
$$y = C\, \mathbf{e}^{(2t)}$$

2. Find the particular solution:

$$C = -6$$

$$y = -6\, \mathbf{e}^{(2t)}$$

3. Sketch the particular solution:



2nd-order, 1 variable: Real eigenvalues

Problem:

$y'' - 2y' - 3y = 0$

$y(0) = 0$

$y'(0) = -4$

1. Find the general solution:

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = -1, 3$

$y = C_1 e^{(-t)} + C_2 e^{(3t)}$

2. Find the particular solution:

$y' = -C_1 e^{(-t)} + 3 C_2 e^{(3t)}$

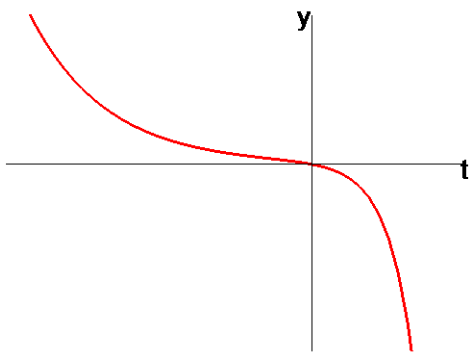
$C_1 + C_2 = 0$

$-C_1 + 3 C_2 = -4$

$C_1 = 1, C_2 = -1$

$y = e^{(-t)} - e^{(3t)}$

3. Sketch the particular solution:



2nd-order, 1 variable: Non-real eigenvalues

Problem:

$$y'' - 2y' + 37y = 0$$

$$y(0) = -1$$

$$y'(0) = 11$$

1. Find the general solution:

$$\lambda^2 - 2\lambda + 37 = 0$$

$$\lambda = 1 + 6I,\ 1 - 6I$$

$$y = \mathbf{e}^t (C_1 \cos(6t) + C_2 \sin(6t))$$

2. Find the particular solution:

$$y' = \mathbf{e}^t (C_1 \cos(6t) + C_2 \sin(6t)) + \mathbf{e}^t (-6 C_1 \sin(6t) + 6 C_2 \cos(6t))$$

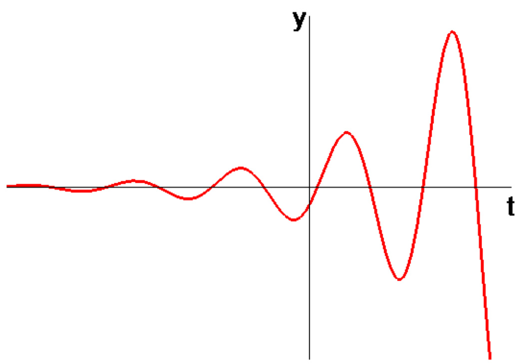
$$C_1 = -1$$

$$C_1 + 6 C_2 = 11$$

$$C_2 = 2$$

$$y = \mathbf{e}^t (-\cos(6t) + 2 \sin(6t))$$

3. Sketch the particular solution:



1st-order, 2 variables: Real eigenvalues

Problem:

$$y' = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

1. Find the general solution:

$$Det \begin{bmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0$$

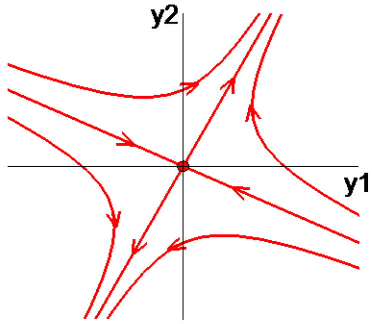
$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2,\ v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\lambda = -3,\ v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$y = C_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \mathbf{e}^{(2\,t)} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mathbf{e}^{(-3\,t)}$$

2. Sketch the general solution:



Saddle

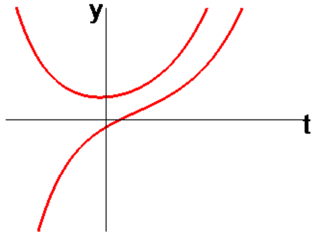
3. Find the particular solution:

$$C_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$C_1 = 1,\ C_2 = -2$$

$$y = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \mathbf{e}^{(2\,t)} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} \mathbf{e}^{(-3\,t)}$$

4. Sketch the particular solution:



Problem:

$$y' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

1. Find the general solution:

$$Det\begin{bmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{bmatrix} = 0$$

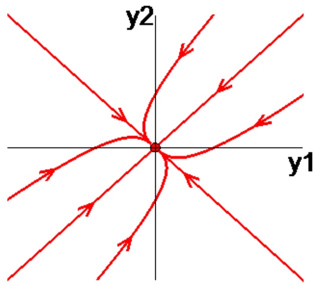
$$\lambda^2 + 4\lambda + 3 = 0$$

$$\lambda = -3,\ v = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\lambda = -1,\ v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$y = C_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{e}^{(-3t)} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathbf{e}^{(-t)}$$

2. Sketch the general solution:



Stable

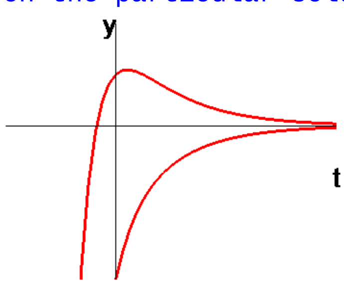
3. Find the particular solution:

$$C_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$C_1 = 1,\ C_2 = -2$$

$$y = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{e}^{(-3t)} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \mathbf{e}^{(-t)}$$

4. Sketch the particular solution:



1st-order 2 variables: Non-real eigenvalues

Problem:

$$y' = \begin{bmatrix} 3 & 5 \\ -4 & -1 \end{bmatrix} y$$

$$y(0) = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

1. Find the general solution:

$$Det \begin{bmatrix} 3-\lambda & 5 \\ -4 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 2\lambda + 17 = 0$$

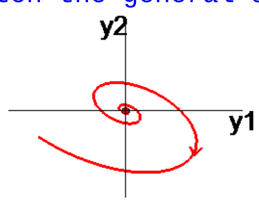
$$\lambda = 1 + 4I, v = \begin{bmatrix} 5 \\ -2+4I \end{bmatrix}$$

$$\lambda = 1 - 4I, v = \begin{bmatrix} 5 \\ -2-4I \end{bmatrix}$$

$$\alpha = 1, \beta = 4, p = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$y = \mathbf{e}^t \left(\left(C_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) \cos(4t) + \left(C_2 \begin{bmatrix} 5 \\ -2 \end{bmatrix} - C_1 \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) \sin(4t) \right)$$

2. Sketch the general solution:



Unstable

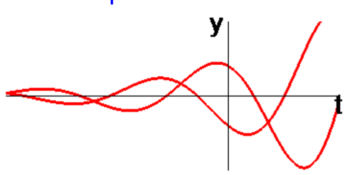
3. Find the particular solution:

$$C_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$C_1 = \frac{4}{5}, C_2 = \frac{-3}{5}$$

$$y = \mathbf{e}^t \left(\begin{bmatrix} 4 \\ -4 \end{bmatrix} \cos(4t) + \begin{bmatrix} -3 \\ -2 \end{bmatrix} \sin(4t) \right)$$

4. Sketch the particular solution:



7. We repeatedly rewrite

$$
\begin{aligned}
y &= C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t} \\
&= C_1 e^{\alpha t+i\beta t} + C_2 e^{\alpha t-i\beta t} \\
&= C_1 e^{\alpha t} e^{i\beta t} + C_2 e^{\alpha t} e^{-i\beta t} \\
&= e^{\alpha t}(C_1 e^{i\beta t} + C_2 e^{-i\beta t}) \\
&= e^{\alpha t}(C_1(\cos\beta t + i\sin\beta t) + C_2(\cos(-\beta t) + i\sin(-\beta t))) \\
&= e^{\alpha t}(C_1(\cos\beta t + i\sin\beta t) + C_2(\cos\beta t - i\sin\beta t)) \\
&= e^{\alpha t}((C_1 + C_2)\cos\beta t + (C_1 - C_2)i\sin\beta t) \\
&= e^{\alpha t}(D_1\cos\beta t + D_2\sin\beta t)
\end{aligned}
$$

By following the convention, we replace D with C, obtaining

$$y = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t)$$