Test 3

1

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -1 \\ 1 & -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

~~1. Compute S = At A~~

$$S = \begin{bmatrix} 14 & 1 \\ 1 & 18 \end{bmatrix}$$

~~2. Cholesky Decomposition~~

$$L = \begin{bmatrix} 3.742 & 0 \\ 0.267 & 4.234 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 3.742 & 0 \\ 0.267 & 4.234 \end{bmatrix} . y = \begin{bmatrix} 12 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 3.742 & 0.267 \\ 0 & 4.234 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} 0.789 \\ 0.956 \end{bmatrix}$$

~~5. Error~~

$$e = \begin{bmatrix} -0.466 \\ -0.323 \\ -0.036 \end{bmatrix}$$

$$||e|| = 0.568$$

2

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -1 \\ 1 & -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} 0.535 & 0.202 \\ -0.802 & -0.186 \\ 0.267 & -0.962 \end{bmatrix}, R = \begin{bmatrix} 3.742 & 0.267 \\ 0.000 & 4.234 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 3.742 & 0.267 \\ 0.000 & 4.234 \end{bmatrix} . x = \begin{bmatrix} 3.207 \\ 4.049 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.789 \\ 0.956 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -0.466 \\ -0.323 \\ -0.036 \end{bmatrix}$$

$$||e|| = 0.568$$

3

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -1 \\ 1 & -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} 0.336 & 0.463 \\ -0.389 & -0.725 \\ -0.858 & 0.510 \end{bmatrix}, \Sigma = \begin{bmatrix} 4.270 & 0.000 \\ 0.000 & 3.710 \end{bmatrix}, V = \begin{bmatrix} 0.230 & 0.973 \\ 0.973 & -0.230 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.789 \\ 0.956 \end{bmatrix}$$

~~4. Error~~

$$e = \begin{bmatrix} -0.466 \\ -0.323 \\ -0.036 \end{bmatrix}$$

$$||e|| = 0.568$$

4

Find the smallest solution of A x = b

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & -4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1. ~~SVD of At~~

$$U = \begin{bmatrix} 0.336 & 0.463 \\ -0.389 & -0.725 \\ -0.858 & 0.510 \end{bmatrix}, \Sigma = \begin{bmatrix} 4.270 & 0.000 \\ 0.000 & 3.710 \end{bmatrix}, V = \begin{bmatrix} 0.230 & 0.973 \\ 0.973 & -0.230 \end{bmatrix}$$

2. SVD-based formula for the smallest solution

$$x = U \Sigma^{-1} V^t b$$

3. Smallest solution

$$x = \begin{bmatrix} 0.235 \\ -0.299 \\ -0.367 \end{bmatrix}$$

5

Compress/Decompress the image A using SVD

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

~~1. SVD of A~~

$$U = \begin{bmatrix} -0.405 & -0.915 \\ -0.915 & 0.405 \end{bmatrix}, \Sigma = \begin{bmatrix} 5.465 & 0.000 \\ 0.000 & 0.366 \end{bmatrix}, V = \begin{bmatrix} -0.576 & 0.817 \\ -0.817 & -0.576 \end{bmatrix}$$

2. Using 1 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.405 \\ -0.915 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 5.465 \end{bmatrix}, V_c = \begin{bmatrix} -0.576 \\ -0.817 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 1.274 & 1.807 \\ 2.879 & 4.085 \end{bmatrix}$$

~~3. Using 2 singular values.~~

~~Compressing~~

$$U_c = \begin{bmatrix} -0.405 & -0.915 \\ -0.915 & 0.405 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 5.465 & 0.000 \\ 0.000 & 0.366 \end{bmatrix}, V_c = \begin{bmatrix} -0.576 & 0.817 \\ -0.817 & -0.576 \end{bmatrix}$$

~~Decompressing~~

$$A_c = \begin{bmatrix} 1.000 & 2.000 \\ 3.000 & 4.000 \end{bmatrix}$$

6

Images: original A and watermark W

$$A = \begin{bmatrix} 1.000 & 2.000 \\ 3.000 & 4.000 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.000 & 0.000 \\ 0.000 & 1.000 \end{bmatrix}$$

1. Embed the watermark

~~a. SVD of A~~

$$U = \begin{bmatrix} -0.405 & -0.915 \\ -0.915 & 0.405 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 5.465 & 0.000 \\ 0.000 & 0.366 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.576 & 0.817 \\ -0.817 & -0.576 \end{bmatrix}$$

b. Embed the watermark into the singular value matrix

$k = 0.100$

$$\Sigma_w = \begin{bmatrix} 5.465 & 0.000 \\ 0.000 & 0.466 \end{bmatrix}$$

c. Construct the watermarked image

$$A_w = \begin{bmatrix} 0.925 & 2.053 \\ 3.033 & 3.977 \end{bmatrix}$$

2. Bad person distorts the watermarked image

a. Filter

$$filter = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 100 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

b. Convolution

$$A_{wd} = \begin{bmatrix} 0.936 & 2.042 \\ 3.042 & 3.967 \end{bmatrix}$$

3. Detect the watermark

a. Recover the watermarked singular values

$$\Sigma_{wd} = \begin{bmatrix} 5.462 & -0.018 \\ -0.002 & 0.457 \end{bmatrix}$$

b. Recover the watermark

$$W_d = \begin{bmatrix} -0.032 & -0.182 \\ -0.016 & 0.910 \end{bmatrix}$$

7. We need to solve

$$A^t A x = A^t b$$

Let

$$A^t A = L L^t$$

Then

$$L L^t x = A^t b$$

Hence we can solve the followings instead

$$L y = A^t b$$
$$L^t x = y$$

8. We need to solve

$$A^t A x = A^t b$$

Let

$$A = QR$$

Then

$$(QR)^t Q R x = (QR)^t b$$
$$R^t Q^t Q R x = R^t Q^t b$$
$$R^t R x = R^t Q^t b$$
$$R x = Q^t b$$

9. We need to solve

$$A^t A x = A^t b$$

Let

$$A = U \Sigma V^t$$

Then

$$(U \Sigma V^t)^t U \Sigma V^t x = (U \Sigma V^t)^t b$$
$$V \Sigma^t U^t U \Sigma V^t x = V \Sigma^t U^t b$$
$$V \Sigma^t \Sigma V^t x = V \Sigma^t U^t b$$
$$\Sigma V^t x = U^t b$$
$$V^t x = \Sigma^{-1} U^t b$$
$$x = (V^t)^{-1} \Sigma^{-1} U^t b$$
$$x = V \Sigma^{-1} U^t b$$

10. We need to compute

$$x = A^t (A A^t)^{-1} b$$

Let

$$A^t = U \Sigma V^t$$

Then

$$
\begin{aligned}
x &= A^t(AA^t)^{-1}b \\
&= U\Sigma V^t((U\Sigma V^t)^t U\Sigma V^t)^{-1}b \\
&= U\Sigma V^t(V\Sigma^t U^t U\Sigma V^t)^{-1}b \\
&= U\Sigma V^t(V\Sigma^2 V^t)^{-1}b \\
&= U\Sigma V^t V^{t-1}\Sigma^{-2}V^{-1}b \\
&= U\Sigma^{-1}V^t b
\end{aligned}
$$