Linear Algebra Homework 3

Hoon Hong

Overconstrained system: Best approximate solution

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 33 & -1 \\ -1 & 19 \end{bmatrix} . x = \begin{bmatrix} -17 \\ -12 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} -0.535 \\ -0.660 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} 0.839 \\ 2.875 \\ 0.120 \end{bmatrix}$$

$$||e|| = 2.998$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \\ 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 14 & 14 \\ 14 & 19 \end{bmatrix} . x = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} -2.443 \\ 1.800 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} -2.486 \\ 2.071 \\ -1.243 \end{bmatrix}$$

$$||e|| = 3.466$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} -2 & -3 \\ -1 & -4 \\ -4 & -1 \\ -4 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 37 & 18 \\ 18 & 27 \end{bmatrix} . x = \begin{bmatrix} -18 \\ 8 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} -0.933 \\ 0.919 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} 1.111 \\ -0.741 \\ 0.815 \\ -1.185 \end{bmatrix}$$

$$||e|| = 1.963$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} -3 & 4 \\ 1 & 4 \\ 3 & -1 \\ -2 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 4 \\ -3 \\ 2 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 23 & -7 \\ -7 & 37 \end{bmatrix} . x = \begin{bmatrix} -12 \\ 19 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} -0.388 \\ 0.440 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} 1.924 \\ -2.627 \\ 1.397 \\ -2.105 \end{bmatrix}$$

$$||e|| = 4.121$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} -3 & 4 & 3 \\ -3 & -2 & -4 \\ 4 & 2 & -1 \\ 3 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 43 & 5 & -10 \\ 5 & 25 & 15 \\ -10 & 15 & 35 \end{bmatrix} . x = \begin{bmatrix} 10 \\ 17 \\ 14 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} 0.239 \\ 0.473 \\ 0.266 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} -0.030 \\ 0.273 \\ 0.637 \\ -0.607 \end{bmatrix}$$

$$||e|| = 0.922$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 2 & -2 \\ 4 & -4 & 1 \\ -1 & -3 & -3 \end{bmatrix}, b = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 27 & -12 & 3 \\ -12 & 30 & -1 \\ 3 & -1 & 18 \end{bmatrix} . x = \begin{bmatrix} 28 \\ 8 \\ 0 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} 1.428 \\ 0.831 \\ -0.192 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} -1.499 \\ -3.381 \\ -1.806 \\ 0.653 \end{bmatrix}$$

$$||e|| = 4.168$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} 3 & -3 & 2 \\ 4 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 49 & -17 & 6 \\ -17 & 13 & -8 \\ 6 & -8 & 22 \end{bmatrix} . x = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} 0.560 \\ 1.318 \\ 0.327 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} 1.379 \\ 2.596 \\ -0.785 \\ -2.098 \\ -3.851 \end{bmatrix}$$

$$||e|| = 5.337$$

Find the best approximate solution of A x = b using normal equation

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 2 & -3 \\ 2 & -4 & -2 \\ -4 & -3 & -1 \\ 3 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

1. Normal Equation

$$\begin{bmatrix} 47 & -2 & 3 \\ -2 & 31 & 5 \\ 3 & 5 & 16 \end{bmatrix} . x = \begin{bmatrix} 10 \\ 0 \\ -19 \end{bmatrix}$$

2. Best approximate solution

$$x = \begin{bmatrix} 0.307 \\ 0.232 \\ -1.318 \end{bmatrix}$$

3. Error

$$e = \begin{bmatrix} -0.994 \\ 0.497 \\ 1.319 \\ -1.606 \\ -1.530 \end{bmatrix}$$

$$||e|| = 2.810$$

Overconstrained system: Best approximate solution using Cholesky

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 33 & -1 \\ -1 & 19 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 5.745 & 0 \\ -0.174 & 4.355 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 5.745 & 0 \\ -0.174 & 4.355 \end{bmatrix} . y = \begin{bmatrix} -17 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} 5.745 & -0.174 \\ 0 & 4.355 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} -0.535 \\ -0.660 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} 0.839 \\ 2.875 \\ 0.120 \end{bmatrix}$$

$$||e|| = 2.998$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \\ 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 14 & 14 \\ 14 & 19 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 3.742 & 0 \\ 3.742 & 2.236 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 3.742 & 0 \\ 3.742 & 2.236 \end{bmatrix} .y = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3.742 & 3.742 \\ 0 & 2.236 \end{bmatrix} .x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} -2.443 \\ 1.800 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} -2.486 \\ 2.071 \\ -1.243 \end{bmatrix}$$

$$||e|| = 3.466$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} -2 & -3 \\ -1 & -4 \\ -4 & -1 \\ -4 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 37 & 18 \\ 18 & 27 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 6.083 & 0 \\ 2.959 & 4.271 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 6.083 & 0 \\ 2.959 & 4.271 \end{bmatrix} . y = \begin{bmatrix} -18 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 6.083 & 2.959 \\ 0 & 4.271 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} -0.933 \\ 0.919 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} 1.111 \\ -0.741 \\ 0.815 \\ -1.185 \end{bmatrix}$$

$$||e|| = 1.963$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} -3 & 4 \\ 1 & 4 \\ 3 & -1 \\ -2 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 4 \\ -3 \\ 2 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 23 & -7 \\ -7 & 37 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 4.796 & 0 \\ -1.460 & 5.905 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 4.796 & 0 \\ -1.460 & 5.905 \end{bmatrix} . y = \begin{bmatrix} -12 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 4.796 & -1.460 \\ 0 & 5.905 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} -0.388 \\ 0.440 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} 1.924 \\ -2.627 \\ 1.397 \\ -2.105 \end{bmatrix}$$

$$||e|| = 4.121$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} -3 & 4 & 3 \\ -3 & -2 & -4 \\ 4 & 2 & -1 \\ 3 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 43 & 5 & -10 \\ 5 & 25 & 15 \\ -10 & 15 & 35 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 6.557 & 0 & 0 \\ 0.762 & 4.942 & 0 \\ -1.525 & 3.271 & 4.688 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 6.557 & 0 & 0 \\ 0.762 & 4.942 & 0 \\ -1.525 & 3.271 & 4.688 \end{bmatrix} . y = \begin{bmatrix} 10 \\ 17 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 6.557 & 0.762 & -1.525 \\ 0 & 4.942 & 3.271 \\ 0 & 0 & 4.688 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} 0.239 \\ 0.473 \\ 0.266 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} -0.030 \\ 0.273 \\ 0.637 \\ -0.607 \end{bmatrix}$$

$$||e|| = 0.922$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 2 & -2 \\ 4 & -4 & 1 \\ -1 & -3 & -3 \end{bmatrix}, b = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 27 & -12 & 3 \\ -12 & 30 & -1 \\ 3 & -1 & 18 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 5.196 & 0 & 0 \\ -2.309 & 4.967 & 0 \\ 0.577 & 0.067 & 4.203 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 5.196 & 0 & 0 \\ -2.309 & 4.967 & 0 \\ 0.577 & 0.067 & 4.203 \end{bmatrix} . y = \begin{bmatrix} 28 \\ 8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5.196 & -2.309 & 0.577 \\ 0 & 4.967 & 0.067 \\ 0 & 0 & 4.203 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} 1.428 \\ 0.831 \\ -0.192 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} -1.499 \\ -3.381 \\ -1.806 \\ 0.653 \end{bmatrix}$$

$$||e|| = 4.168$$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} 3 & -3 & 2 \\ 4 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 49 & -17 & 6 \\ -17 & 13 & -8 \\ 6 & -8 & 22 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 7.000 & 0 & 0 \\ -2.429 & 2.665 & 0 \\ 0.857 & -2.221 & 4.041 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 7.000 & 0 & 0 \\ -2.429 & 2.665 & 0 \\ 0.857 & -2.221 & 4.041 \end{bmatrix} . y = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7.000 & -2.429 & 0.857 \\ 0 & 2.665 & -2.221 \\ 0 & 0 & 4.041 \end{bmatrix} . x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} 0.560 \\ 1.318 \\ 0.327 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} 1.379 \\ 2.596 \\ -0.785 \\ -2.098 \\ -3.851 \end{bmatrix}$$

$||e|| = 5.337$

Find the best approximate solution of A x = b using Cholesky

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 2 & -3 \\ 2 & -4 & -2 \\ -4 & -3 & -1 \\ 3 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

1. Compute S = At A

$$S = \begin{bmatrix} 47 & -2 & 3 \\ -2 & 31 & 5 \\ 3 & 5 & 16 \end{bmatrix}$$

2. Cholesky Decomposition

$$L = \begin{bmatrix} 6.856 & 0 & 0 \\ -0.292 & 5.560 & 0 \\ 0.438 & 0.922 & 3.868 \end{bmatrix}$$

3. Cholesky-based formula for the best approximate solution

$$\begin{bmatrix} 6.856 & 0 & 0 \\ -0.292 & 5.560 & 0 \\ 0.438 & 0.922 & 3.868 \end{bmatrix} .y = \begin{bmatrix} 10 \\ 0 \\ -19 \end{bmatrix}$$

$$\begin{bmatrix} 6.856 & -0.292 & 0.438 \\ 0 & 5.560 & 0.922 \\ 0 & 0 & 3.868 \end{bmatrix} .x = y$$

4. Best approximate solution

$$x = \begin{bmatrix} 0.307 \\ 0.232 \\ -1.318 \end{bmatrix}$$

5. Error

$$e = \begin{bmatrix} -0.994 \\ 0.497 \\ 1.319 \\ -1.606 \\ -1.530 \end{bmatrix}$$

$||e|| = 2.810$

Overconstrained system: Best approximate solution using QR

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} 0.696 & -0.661 \\ -0.174 & 0.223 \\ -0.696 & -0.717 \end{bmatrix}, R = \begin{bmatrix} 5.745 & -0.174 \\ 0 & 4.355 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 5.745 & -0.174 \\ 0 & 4.355 \end{bmatrix} . x = \begin{bmatrix} -2.959 \\ -2.873 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.535 \\ -0.660 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 0.839 \\ 2.875 \\ 0.120 \end{bmatrix}$$

$$||e|| = 2.998$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \\ 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} 0.535 & 0.447 \\ 0.802 & 1.247\ 10^{-10} \\ 0.267 & -0.894 \end{bmatrix}, R = \begin{bmatrix} 3.742 & 3.742 \\ 0 & 2.236 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 3.742 & 3.742 \\ 0 & 2.236 \end{bmatrix} . x = \begin{bmatrix} -2.405 \\ 4.025 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} -2.443 \\ 1.800 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -2.486 \\ 2.071 \\ -1.243 \end{bmatrix}$$

$$||e|| = 3.466$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} -2 & -3 \\ -1 & -4 \\ -4 & -1 \\ -4 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} -0.329 & -0.475 \\ -0.164 & -0.823 \\ -0.658 & 0.221 \\ -0.658 & 0.221 \end{bmatrix}, R = \begin{bmatrix} 6.083 & 2.959 \\ 0 & 4.271 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 6.083 & 2.959 \\ 0 & 4.271 \end{bmatrix} .x = \begin{bmatrix} -2.959 \\ 3.923 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.933 \\ 0.919 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.111 \\ -0.741 \\ 0.815 \\ -1.185 \end{bmatrix}$$

$$||e|| = 1.963$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} -3 & 4 \\ 1 & 4 \\ 3 & -1 \\ -2 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 4 \\ -3 \\ 2 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} -0.626 & 0.523 \\ 0.209 & 0.729 \\ 0.626 & -0.015 \\ -0.417 & -0.442 \end{bmatrix}, R = \begin{bmatrix} 4.796 & -1.460 \\ 0 & 5.905 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 4.796 & -1.460 \\ 0 & 5.905 \end{bmatrix} . x = \begin{bmatrix} -2.502 \\ 2.599 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.388 \\ 0.440 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.924 \\ -2.627 \\ 1.397 \\ -2.105 \end{bmatrix}$$

$$||e|| = 4.121$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} -3 & 4 & 3 \\ -3 & -2 & -4 \\ 4 & 2 & -1 \\ 3 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} -0.457 & 0.880 & -0.123 \\ -0.457 & -0.334 & -0.769 \\ 0.610 & 0.311 & -0.232 \\ 0.457 & 0.132 & -0.583 \end{bmatrix}, R = \begin{bmatrix} 6.557 & 0.762 & -1.525 \\ 0 & 4.942 & 3.271 \\ 0 & 0 & 4.688 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 6.557 & 0.762 & -1.525 \\ 0 & 4.942 & 3.271 \\ 0 & 0 & 4.688 \end{bmatrix} . x = \begin{bmatrix} 1.525 \\ 3.205 \\ 1.246 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.239 \\ 0.473 \\ 0.266 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -0.030 \\ 0.273 \\ 0.637 \\ -0.607 \end{bmatrix}$$

$$||e|| = 0.922$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 2 & -2 \\ 4 & -4 & 1 \\ -1 & -3 & -3 \end{bmatrix}, b = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} -0.577 & -0.470 & 0.563 \\ -0.192 & 0.313 & -0.454 \\ 0.770 & -0.447 & 0.139 \\ -0.192 & -0.694 & -0.676 \end{bmatrix}, R = \begin{bmatrix} 5.196 & -2.309 & 0.577 \\ 0 & 4.967 & 0.067 \\ 0 & 0 & 4.203 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 5.196 & -2.309 & 0.577 \\ 0 & 4.967 & 0.067 \\ 0 & 0 & 4.203 \end{bmatrix} .x = \begin{bmatrix} 5.389 \\ 4.116 \\ -0.806 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} 1.428 \\ 0.831 \\ -0.192 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -1.499 \\ -3.381 \\ -1.806 \\ 0.653 \end{bmatrix}$$

$$||e|| = 4.168$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} 3 & -3 & 2 \\ 4 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} 0.429 & -0.735 & 1.449\ 10^{-10} \\ 0.571 & 0.146 & -0.289 \\ -0.286 & -0.636 & 0.206 \\ 0.571 & 0.146 & 0.701 \\ 0.286 & -0.115 & -0.619 \end{bmatrix}, R = \begin{bmatrix} 7.000 & -2.429 & 0.857 \\ 0 & 2.665 & -2.221 \\ 0 & 0 & 4.041 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 7.000 & -2.429 & 0.857 \\ 0 & 2.665 & -2.221 \\ 0 & 0 & 4.041 \end{bmatrix} .x = \begin{bmatrix} 1.000 \\ 2.787 \\ 1.320 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.560 \\ 1.318 \\ 0.327 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.379 \\ 2.596 \\ -0.785 \\ -2.098 \\ -3.851 \end{bmatrix}$$

$$||e|| = 5.337$$

Find the best approximate solution of A x = b using QR

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 2 & -3 \\ 2 & -4 & -2 \\ -4 & -3 & -1 \\ 3 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

1. QR Decomposition

$$Q = \begin{bmatrix} 0.438 & -0.157 & -0.271 \\ -0.438 & 0.337 & -0.806 \\ 0.292 & -0.704 & -0.382 \\ -0.583 & -0.570 & -0.057 \\ 0.438 & 0.203 & -0.356 \end{bmatrix}, R = \begin{bmatrix} 6.856 & -0.292 & 0.438 \\ 0 & 5.560 & 0.922 \\ 0 & 0 & 3.868 \end{bmatrix}$$

2. QR-based formula for the best approximate solution

$$\begin{bmatrix} 6.856 & -0.292 & 0.438 \\ 0 & 5.560 & 0.922 \\ 0 & 0 & 3.868 \end{bmatrix} .x = \begin{bmatrix} 1.459 \\ 0.077 \\ -5.096 \end{bmatrix}$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.307 \\ 0.232 \\ -1.318 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -0.994 \\ 0.497 \\ 1.319 \\ -1.606 \\ -1.530 \end{bmatrix}$$

$$||e|| = 2.810$$

Overconstrained system: Best approximate solution using SVD

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.731 & 0.623 \\ 0.186 & -0.213 \\ 0.657 & 0.753 \end{bmatrix}, \Sigma = \begin{bmatrix} 5.751 & 0.000 \\ 0.000 & 4.351 \end{bmatrix}, V = \begin{bmatrix} -0.997 & -0.071 \\ 0.071 & -0.997 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.535 \\ -0.660 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 0.839 \\ 2.875 \\ 0.120 \end{bmatrix}$$

$$||e|| = 2.998$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \\ 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.647 & 0.260 \\ -0.762 & -0.248 \\ 0.023 & -0.933 \end{bmatrix}, \Sigma = \begin{bmatrix} 5.543 & 0.000 \\ 0.000 & 1.509 \end{bmatrix}, V = \begin{bmatrix} -0.642 & -0.767 \\ -0.767 & 0.642 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} -2.443 \\ 1.800 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -2.486 \\ 2.071 \\ -1.243 \end{bmatrix}$$

$$||e|| = 3.466$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} -2 & -3 \\ -1 & -4 \\ -4 & -1 \\ -4 & -1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.479 & -0.323 \\ -0.452 & -0.707 \\ -0.532 & 0.445 \\ -0.532 & 0.445 \end{bmatrix}, \Sigma = \begin{bmatrix} 7.119 & 0.000 \\ 0.000 & 3.649 \end{bmatrix}, V = \begin{bmatrix} 0.796 & -0.605 \\ 0.605 & 0.796 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.933 \\ 0.919 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.111 \\ -0.741 \\ 0.815 \\ -1.185 \end{bmatrix}$$

$$||e|| = 1.963$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} -3 & 4 \\ 1 & 4 \\ 3 & -1 \\ -2 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 4 \\ -3 \\ 2 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.767 & 0.277 \\ -0.524 & -0.547 \\ 0.328 & -0.533 \\ 0.171 & 0.583 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.317 & 0.000 \\ 0.000 & 4.483 \end{bmatrix}, V = \begin{bmatrix} 0.383 & -0.924 \\ -0.924 & -0.383 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} -0.388 \\ 0.440 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.924 \\ -2.627 \\ 1.397 \\ -2.105 \end{bmatrix}$$

$$||e|| = 4.121$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} -3 & 4 & 3 \\ -3 & -2 & -4 \\ 4 & 2 & -1 \\ 3 & 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.719 & -0.280 & 0.635 \\ 0.161 & 0.772 & 0.538 \\ 0.409 & -0.534 & 0.264 \\ 0.538 & -0.199 & 0.487 \end{bmatrix}, \Sigma = \begin{bmatrix} 7.137 & 0.000 & 0.000 \\ 0.000 & 6.405 & 0.000 \\ 0.000 & 0.000 & 3.323 \end{bmatrix}, V = \begin{bmatrix} 0.690 & -0.658 & -0.302 \\ -0.258 & -0.614 & 0.746 \\ -0.676 & -0.437 & -0.593 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.239 \\ 0.473 \\ 0.266 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -0.030 \\ 0.273 \\ 0.637 \\ -0.607 \end{bmatrix}$$

$$||e|| = 0.922$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 2 & -2 \\ 4 & -4 & 1 \\ -1 & -3 & -3 \end{bmatrix}, b = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.158 & -0.033 & -0.919 \\ -0.372 & 0.254 & 0.372 \\ 0.895 & -0.101 & 0.019 \\ 0.187 & 0.961 & -0.128 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.397 & 0.000 & 0.000 \\ 0.000 & 4.329 & 0.000 \\ 0.000 & 0.000 & 3.917 \end{bmatrix}, V$$

$$= \begin{bmatrix} 0.663 & -0.352 & 0.661 \\ -0.739 & -0.448 & 0.503 \\ 0.119 & -0.822 & -0.557 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} 1.428 \\ 0.831 \\ -0.192 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -1.499 \\ -3.381 \\ -1.806 \\ 0.653 \end{bmatrix}$$

$$||e|| = 4.168$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} 3 & -3 & 2 \\ 4 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.565 & 0.316 & -0.552 \\ -0.490 & -0.437 & 0.001 \\ 0.123 & 0.582 & -0.417 \\ -0.613 & 0.355 & 0.581 \\ -0.224 & -0.494 & -0.429 \end{bmatrix}, \Sigma = \begin{bmatrix} 7.604 & 0.000 & 0.000 \\ 0.000 & 4.651 & 0.000 \\ 0.000 & 0.000 & 2.132 \end{bmatrix}, V = \begin{bmatrix} -0.894 & -0.330 & 0.303 \\ 0.381 & -0.205 & 0.901 \\ -0.235 & 0.922 & 0.309 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.560 \\ 1.318 \\ 0.327 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} 1.379 \\ 2.596 \\ -0.785 \\ -2.098 \\ -3.851 \end{bmatrix}$$

$$||e|| = 5.337$$

Find the best approximate solution of A x = b using SVD

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 2 & -3 \\ 2 & -4 & -2 \\ -4 & -3 & -1 \\ 3 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

1. SVD of A

$$U = \begin{bmatrix} -0.435 & -0.184 & 0.258 \\ 0.495 & 0.139 & 0.831 \\ -0.321 & -0.750 & 0.248 \\ 0.546 & -0.602 & -0.093 \\ -0.407 & 0.150 & 0.414 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.887 & 0.000 & 0.000 \\ 0.000 & 5.696 & 0.000 \\ 0.000 & 0.000 & 3.758 \end{bmatrix}, V = \begin{bmatrix} -0.992 & 0.068 & 0.105 \\ 0.097 & 0.951 & 0.294 \\ -0.079 & 0.302 & -0.950 \end{bmatrix}$$

2. SVD-based formula for the best approximate solution

$$x = V \Sigma^{-1} U^t b$$

3. Best approximate solution

$$x = \begin{bmatrix} 0.307 \\ 0.232 \\ -1.318 \end{bmatrix}$$

4. Error

$$e = \begin{bmatrix} -0.994 \\ 0.497 \\ 1.319 \\ -1.606 \\ -1.530 \end{bmatrix}$$

$$||e|| = 2.810$$

Function Fitting

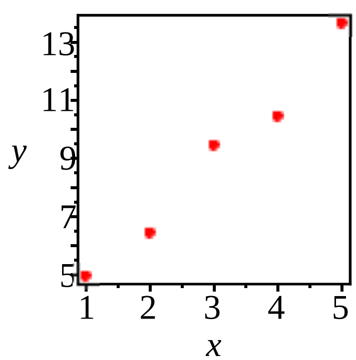
Fit the model to the data:

Model

$$y = c_1 x + c_2$$

Data

x	y
1.000	4.950
2.000	6.440
3.000	9.450
4.000	10.450
5.000	13.650



1. Equation

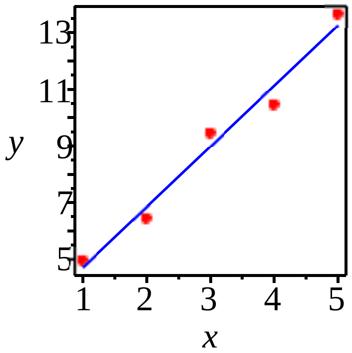
$$\begin{bmatrix} 1.000 & 1.000 \\ 2.000 & 1.000 \\ 3.000 & 1.000 \\ 4.000 & 1.000 \\ 5.000 & 1.000 \end{bmatrix} . c = \begin{bmatrix} 4.950 \\ 6.440 \\ 9.450 \\ 10.450 \\ 13.650 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} 2.141 \\ 2.565 \end{bmatrix}$$

3. Best Fit

$$y = 2.141\, x + 2.565$$



4. Error

$$e = \begin{bmatrix} -0.244 \\ 0.407 \\ -0.462 \\ 0.679 \\ -0.380 \end{bmatrix}$$

$$||e|| = 1.022$$

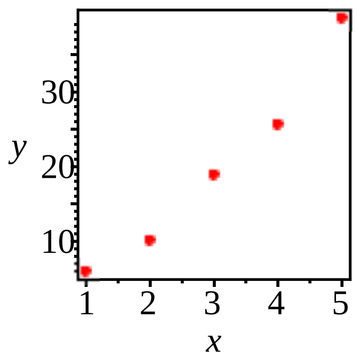
Fit the model to the data:

Model

$$y = c_1 x^2 + c_2 x + c_3$$

Data

x	y
1.000	5.940
2.000	10.120
3.000	18.900
4.000	25.650
5.000	39.900



1. Equation

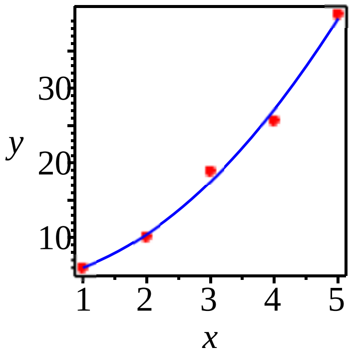
$$\begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 4.000 & 2.000 & 1.000 \\ 9.000 & 3.000 & 1.000 \\ 16.000 & 4.000 & 1.000 \\ 25.000 & 5.000 & 1.000 \end{bmatrix} . c = \begin{bmatrix} 5.940 \\ 10.120 \\ 18.900 \\ 25.650 \\ 39.900 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} 1.294 \\ 0.584 \\ 4.122 \end{bmatrix}$$

3. Best Fit

$$y = 1.294\, x^2 + 0.584\, x + 4.122$$



4. Error

$$e = \begin{bmatrix} 0.059 \\ 0.343 \\ -1.385 \\ 1.503 \\ -0.521 \end{bmatrix}$$

$$||e|| = 2.138$$

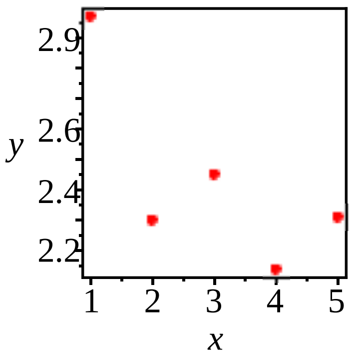
Fit the model to the data:

Model

$$y = \frac{c_1}{x} + c_2$$

Data

x	y
1.000	2.970
2.000	2.300
3.000	2.450
4.000	2.138
5.000	2.310



1. Equation

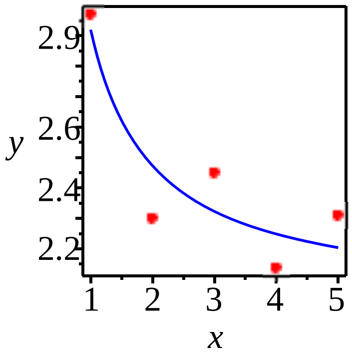
$$\begin{bmatrix} 1.000 & 1.000 \\ 0.500 & 1.000 \\ 0.333 & 1.000 \\ 0.250 & 1.000 \\ 0.200 & 1.000 \end{bmatrix} . c = \begin{bmatrix} 2.970 \\ 2.300 \\ 2.450 \\ 2.138 \\ 2.310 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} 0.895 \\ 2.025 \end{bmatrix}$$

3. Best Fit

$$y = \frac{0.895}{x} + 2.025$$



4. Error

$$e = \begin{bmatrix} -0.050 \\ 0.172 \\ -0.127 \\ 0.111 \\ -0.106 \end{bmatrix}$$

$$||e|| = 0.268$$

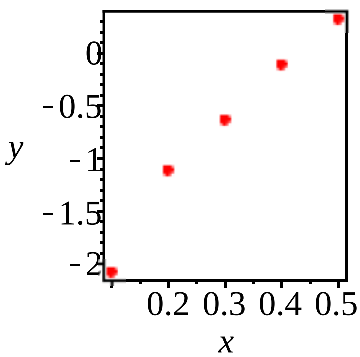
Fit the model to the data:

Model

$$y = c_1 x + c_2 \ln(x)$$

Data

x	y
0.100	-2.082
0.200	-1.113
0.300	-0.634
0.400	-0.110
0.500	0.322



1. Equation

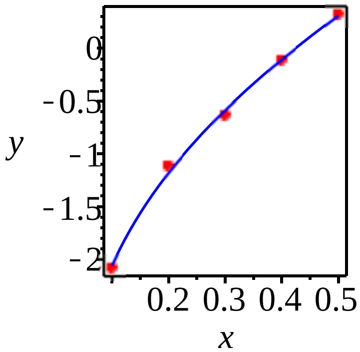
$$\begin{bmatrix} 0.100 & -2.303 \\ 0.200 & -1.609 \\ 0.300 & -1.204 \\ 0.400 & -0.916 \\ 0.500 & -0.693 \end{bmatrix} . c = \begin{bmatrix} -2.082 \\ -1.113 \\ -0.634 \\ -0.110 \\ 0.322 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} 1.981 \\ 0.980 \end{bmatrix}$$

3. Best Fit

$$y = 1.981\,x + 0.980 \ln(x)$$



4. Error

$$e = \begin{bmatrix} 0.023 \\ -0.068 \\ 0.049 \\ 0.005 \\ -0.011 \end{bmatrix}$$

$$||e|| = 0.088$$

Equation Fitting

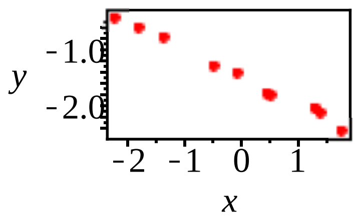
Fit the model to the data:

Model

$$c_1 x + c_2 y + c_3 = 0$$

Data

x	y
0.461	-1.783
0.526	-1.814
1.766	-2.453
-2.214	-0.437
1.397	-2.131
-0.475	-1.298
-1.352	-0.786
1.304	-2.045
-0.061	-1.419
-1.792	-0.611



1. Equation

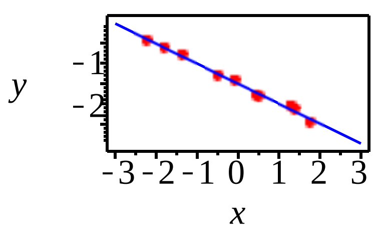
$$\begin{bmatrix} 0.461 & -1.784 & 1.000 \\ 0.527 & -1.814 & 1.000 \\ 1.766 & -2.453 & 1.000 \\ -2.214 & -0.437 & 1.000 \\ 1.397 & -2.131 & 1.000 \\ -0.475 & -1.298 & 1.000 \\ -1.352 & -0.786 & 1.000 \\ 1.304 & -2.045 & 1.000 \\ -0.061 & -1.419 & 1.000 \\ -1.792 & -0.611 & 1.000 \\ 1 & 1 & 1 \end{bmatrix} . c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} 0.164 \\ 0.333 \\ 0.499 \end{bmatrix}$$

3. Best Fit

$$0.164\,x + 0.333\,y + 0.499 = 0$$



4. Error

$$e = \begin{bmatrix} -0.019 \\ -0.018 \\ -0.027 \\ -0.009 \\ 0.019 \\ -0.011 \\ 0.016 \\ 0.033 \\ 0.017 \\ 0.002 \\ -0.004 \end{bmatrix}$$

$$||e|| = 0.060$$

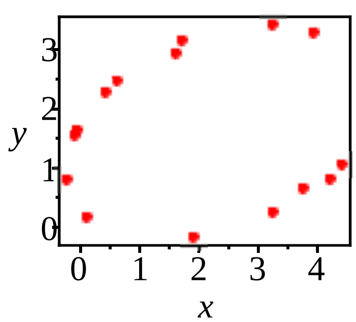
Fit the model to the data:

Model

$$c_1 x^2 + c_2 y^2 + c_3 x y + c_4 x + c_5 y + c_6 = 0$$

Data

x	y
4.417	1.053
-0.050	1.635
0.432	2.278
3.764	0.652
1.721	3.154
1.620	2.936
4.219	0.810
3.255	0.248
1.914	-0.173
0.115	0.168
-0.221	0.802
-0.083	1.552
3.250	3.422
3.942	3.286
0.625	2.472



1. Equation

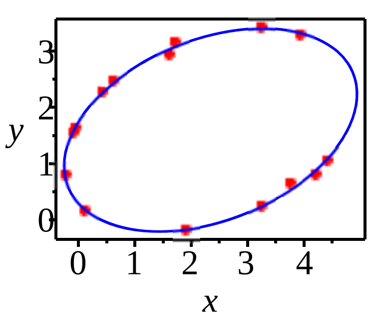
$$\begin{bmatrix} 19.506 & 1.109 & 4.651 & 4.417 & 1.053 & 1.000 \\ 0.002 & 2.674 & -0.081 & -0.050 & 1.635 & 1.000 \\ 0.186 & 5.191 & 0.984 & 0.432 & 2.278 & 1.000 \\ 14.170 & 0.425 & 2.455 & 3.764 & 0.652 & 1.000 \\ 2.962 & 9.945 & 5.427 & 1.721 & 3.154 & 1.000 \\ 2.623 & 8.623 & 4.756 & 1.620 & 2.936 & 1.000 \\ 17.801 & 0.656 & 3.417 & 4.219 & 0.810 & 1.000 \\ 10.595 & 0.062 & 0.808 & 3.255 & 0.248 & 1.000 \\ 3.663 & 0.030 & -0.331 & 1.914 & -0.173 & 1.000 \\ 0.013 & 0.028 & 0.019 & 0.115 & 0.168 & 1.000 \\ 0.049 & 0.643 & -0.177 & -0.221 & 0.802 & 1.000 \\ 0.007 & 2.409 & -0.129 & -0.083 & 1.552 & 1.000 \\ 10.563 & 11.710 & 11.122 & 3.250 & 3.422 & 1.000 \\ 15.543 & 10.800 & 12.956 & 3.942 & 3.286 & 1.000 \\ 0.391 & 6.111 & 1.545 & 0.625 & 2.472 & 1.000 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} . c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} -0.221 \\ -0.459 \\ 0.226 \\ 0.675 \\ 0.939 \\ -0.236 \end{bmatrix}$$

3. Best Fit

$$-0.221\,x^2 - 0.459\,y^2 + 0.226\,x\,y + 0.675\,x + 0.939\,y - 0.236 = 0$$



4. Error

$$e = \begin{bmatrix} -0.037 \\ 0.019 \\ -0.008 \\ 0.144 \\ -0.110 \\ 0.149 \\ -0.092 \\ 0.007 \\ -0.004 \\ -0.013 \\ 0.022 \\ 0.028 \\ -0.031 \\ 0.040 \\ -0.037 \\ -0.076 \end{bmatrix}$$

$$||e|| = 0.276$$

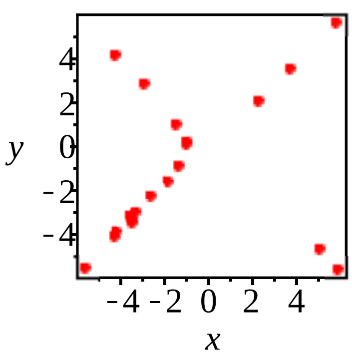
Fit the model to the data:

Model

$$c_1 x^2 + c_2 y^2 + c_3 x y + c_4 x + c_5 y + c_6 = 0$$

Data

x	y
-3.321	-3.004
3.726	3.544
-3.502	-3.331
-0.995	0.124
-1.006	0.233
-3.555	-3.154
-4.183	-3.878
5.068	-4.673
-5.615	-5.533
-4.235	4.169
2.286	2.070
-3.463	-3.466
-1.843	-1.583
5.900	-5.612
-1.462	1.003
-1.346	-0.886
-2.638	-2.264
-2.926	2.866
-4.268	-4.092
5.817	5.645



1. Equation

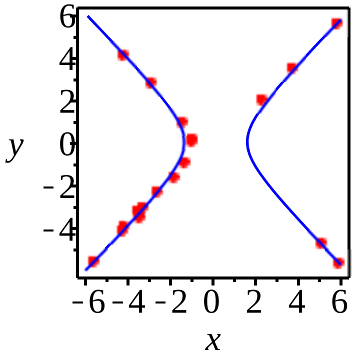
$$\begin{bmatrix} 11.029 & 9.026 & 9.977 & -3.321 & -3.004 & 1.000 \\ 13.883 & 12.563 & 13.207 & 3.726 & 3.544 & 1.000 \\ 12.266 & 11.093 & 11.665 & -3.502 & -3.331 & 1.000 \\ 0.991 & 0.015 & -0.123 & -0.995 & 0.124 & 1.000 \\ 1.012 & 0.054 & -0.234 & -1.006 & 0.233 & 1.000 \\ 12.635 & 9.949 & 11.212 & -3.555 & -3.154 & 1.000 \\ 17.495 & 15.039 & 16.221 & -4.183 & -3.878 & 1.000 \\ 25.681 & 21.833 & -23.679 & 5.068 & -4.673 & 1.000 \\ 31.533 & 30.618 & 31.072 & -5.615 & -5.533 & 1.000 \\ 17.932 & 17.379 & -17.653 & -4.235 & 4.169 & 1.000 \\ 5.224 & 4.285 & 4.731 & 2.286 & 2.070 & 1.000 \\ 11.990 & 12.014 & 12.002 & -3.463 & -3.466 & 1.000 \\ 3.396 & 2.507 & 2.918 & -1.843 & -1.583 & 1.000 \\ 34.805 & 31.492 & -33.107 & 5.900 & -5.612 & 1.000 \\ 2.136 & 1.007 & -1.467 & -1.462 & 1.003 & 1.000 \\ 1.813 & 0.785 & 1.193 & -1.346 & -0.886 & 1.000 \\ 6.962 & 5.126 & 5.974 & -2.638 & -2.264 & 1.000 \\ 8.563 & 8.212 & -8.385 & -2.926 & 2.866 & 1.000 \\ 18.219 & 16.742 & 17.465 & -4.268 & -4.092 & 1.000 \\ 33.834 & 31.864 & 32.834 & 5.817 & 5.645 & 1.000 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} . c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Least Square

$$c = \begin{bmatrix} -0.127 \\ 0.123 \\ 0.000 \\ 0.031 \\ -0.020 \\ 0.277 \end{bmatrix}$$

3. Best Fit

$$-0.127\, x^2 + 0.123\, y^2 + 0.000\, x\, y + 0.031\, x - 0.020\, y + 0.277 = 0$$



4. Error

$$e = \begin{bmatrix} -0.050 \\ 0.113 \\ 0.049 \\ 0.120 \\ 0.120 \\ -0.144 \\ -0.137 \\ -0.037 \\ -0.005 \\ -0.067 \\ 0.174 \\ 0.202 \\ 0.130 \\ 0.042 \\ 0.065 \\ 0.120 \\ -0.010 \\ 0.057 \\ -0.017 \\ -0.010 \\ -0.715 \end{bmatrix}$$

$||e|| = 0.846$

Underconstrained system: Smallest solution

Find the smallest solution of A x = b

$$A = \begin{bmatrix} 4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \end{bmatrix}$$

1. Formula for the smallest solution

$$x = A^t \left(A A^t\right)^{-1} b$$

2. Smallest solution

$$x = \begin{bmatrix} -0.160 \\ 0.120 \end{bmatrix}$$

Find the smallest solution of A x = b

$$A = \begin{bmatrix} 4 & -3 & -1 \\ 1 & -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

1. Formula for the smallest solution

$$x = A^t \left(A A^t\right)^{-1} b$$

2. Smallest solution

$$x = \begin{bmatrix} 0.206 \\ 0.454 \\ 0.463 \end{bmatrix}$$

Find the smallest solution of A x = b

$$A = \begin{bmatrix} 4 & -3 & -1 & 1 \\ -4 & -3 & -1 & -3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

1. Formula for the smallest solution

$$x = A^t \left(A A^t\right)^{-1} b$$

2. Smallest solution

$$x = \begin{bmatrix} 0.315 \\ -0.861 \\ -0.287 \\ -0.130 \end{bmatrix}$$

Find the smallest solution of A x = b

$$A = \begin{bmatrix} 4 & -3 & -1 & 1 \\ -4 & -3 & -1 & -3 \\ 4 & 2 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

1. Formula for the smallest solution

$$x = A^t \left(A\,A^t\right)^{-1} b$$

2. Smallest solution

$$x = \begin{bmatrix} 0.279 \\ -0.555 \\ -0.277 \\ -0.058 \end{bmatrix}$$

Underconstrained system: Smallest solution using SVD

Find the smallest solution of A x = b using SVD

$$A = \begin{bmatrix} 4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \end{bmatrix}$$

1. SVD of At

$$U = \begin{bmatrix} -0.800 \\ 0.600 \end{bmatrix}, \Sigma = \begin{bmatrix} 5.000 \end{bmatrix}, V = \begin{bmatrix} -1.000 \end{bmatrix}$$

2. SVD-based formula for the smallest solution

$$x = U \Sigma^{-1} V^t b$$

3. Smallest solution

$$x = \begin{bmatrix} -0.160 \\ 0.120 \end{bmatrix}$$

Find the smallest solution of A x = b using SVD

$$A = \begin{bmatrix} 4 & -3 & -1 \\ 1 & -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

1. SVD of At

$$U = \begin{bmatrix} -0.527 & -0.802 \\ 0.738 & -0.267 \\ 0.422 & -0.535 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.708 & 0.000 \\ 0.000 & 2.646 \end{bmatrix}, V = \begin{bmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

2. SVD-based formula for the smallest solution

$$x = U \Sigma^{-1} V^t b$$

3. Smallest solution

$$x = \begin{bmatrix} 0.206 \\ 0.454 \\ 0.463 \end{bmatrix}$$

Find the smallest solution of A x = b using SVD

$$A = \begin{bmatrix} 4 & -3 & -1 & 1 \\ -4 & -3 & -1 & -3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

1. SVD of At

$$U = \begin{bmatrix} -0.866 & 0.255 \\ -0.138 & -0.902 \\ -0.046 & -0.301 \\ -0.479 & -0.173 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.391 & 0.000 \\ 0.000 & 4.599 \end{bmatrix}, V = \begin{bmatrix} -0.545 & 0.838 \\ 0.838 & 0.545 \end{bmatrix}$$

2. SVD-based formula for the smallest solution

$$x = U \Sigma^{-1} V^t b$$

3. Smallest solution

$$x = \begin{bmatrix} 0.315 \\ -0.861 \\ -0.287 \\ -0.130 \end{bmatrix}$$

Find the smallest solution of A x = b using SVD

$$A = \begin{bmatrix} 4 & -3 & -1 & 1 \\ -4 & -3 & -1 & -3 \\ 4 & 2 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

1. SVD of At

$$U = \begin{bmatrix} -0.758 & 0.478 & 0.114 \\ -0.297 & -0.802 & 0.456 \\ -0.288 & -0.357 & -0.880 \\ -0.505 & -0.041 & 0.062 \end{bmatrix}, \Sigma = \begin{bmatrix} 8.623 & 0.000 & 0.000 \\ 0.000 & 4.815 & 0.000 \\ 0.000 & 0.000 & 1.568 \end{bmatrix}, V$$

$$= \begin{bmatrix} -0.273 & 0.962 & 0.019 \\ 0.664 & 0.203 & -0.720 \\ -0.696 & -0.184 & -0.694 \end{bmatrix}$$

2. SVD-based formula for the smallest solution

$$x = U \Sigma^{-1} V^t b$$

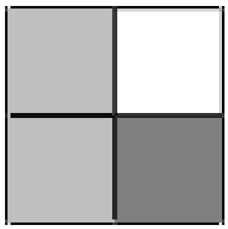
3. Smallest solution

$$x = \begin{bmatrix} 0.279 \\ -0.555 \\ -0.277 \\ -0.058 \end{bmatrix}$$

Image Compression

Compress/Decompress the image A using SVD

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$$



1. SVD of A

$$U = \begin{bmatrix} -0.816 & -0.578 \\ -0.578 & 0.816 \end{bmatrix}, \Sigma = \begin{bmatrix} 6.085 & 0.000 \\ 0.000 & 0.986 \end{bmatrix}, V = \begin{bmatrix} -0.687 & 0.726 \\ -0.726 & -0.687 \end{bmatrix}$$

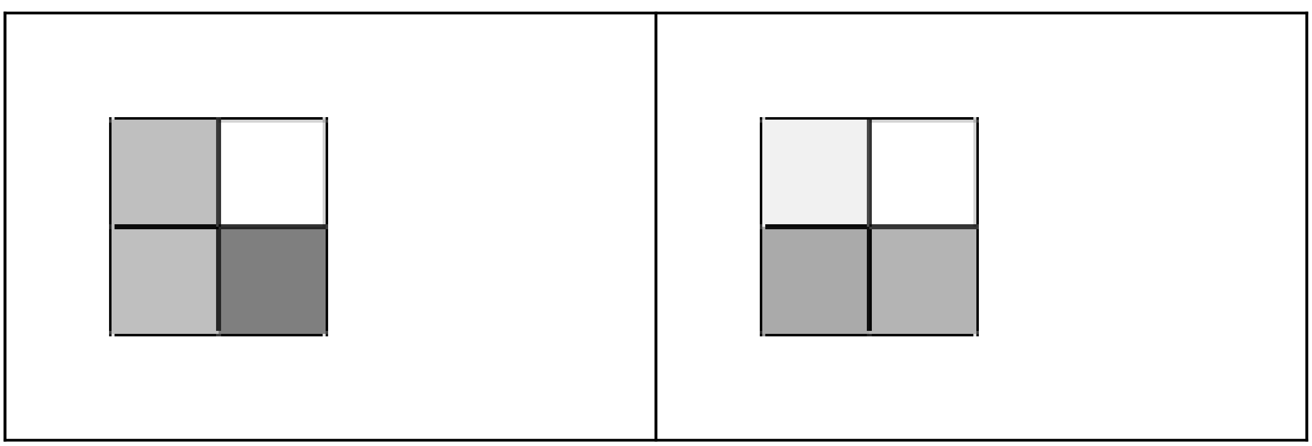
2. Using 1 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.816 \\ -0.578 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 6.085 \end{bmatrix}, V_c = \begin{bmatrix} -0.687 \\ -0.726 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 3.414 & 3.609 \\ 2.415 & 2.553 \end{bmatrix}$$



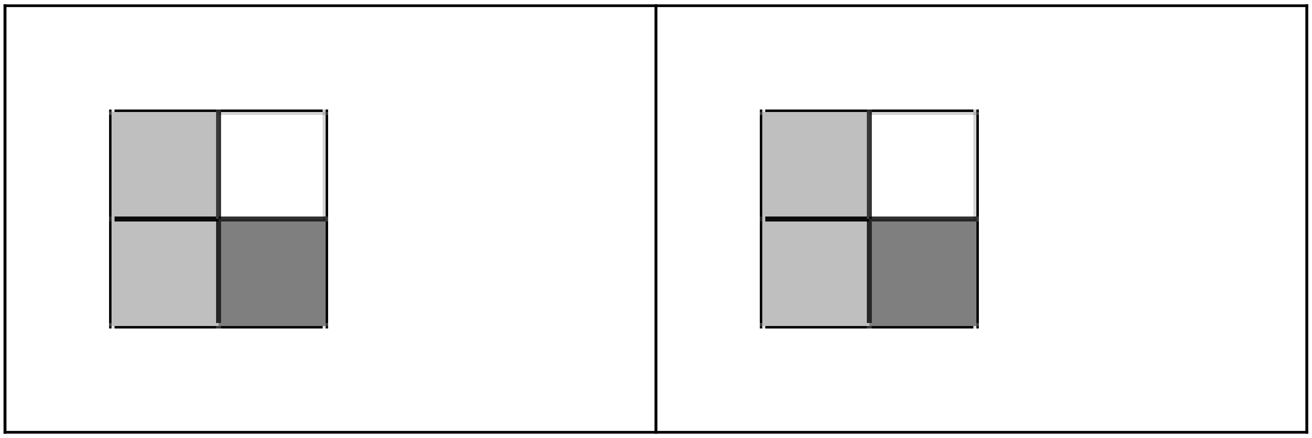
3. Using 2 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.816 & -0.578 \\ -0.578 & 0.816 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 6.085 & 0.000 \\ 0.000 & 0.986 \end{bmatrix}, V_c = \begin{bmatrix} -0.687 & 0.726 \\ -0.726 & -0.687 \end{bmatrix}$$

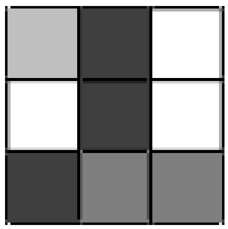
Decompressing

$$A_c = \begin{bmatrix} 3.000 & 4.000 \\ 3.000 & 2.000 \end{bmatrix}$$



Compress/Decompress the image A using SVD

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$$



1. SVD of A

$$U = \begin{bmatrix} -0.630 & -0.077 & -0.773 \\ -0.707 & -0.356 & 0.611 \\ -0.322 & 0.931 & 0.170 \end{bmatrix}, \Sigma = \begin{bmatrix} 8.075 & 0.000 & 0.000 \\ 0.000 & 1.608 & 0.000 \\ 0.000 & 0.000 & 0.462 \end{bmatrix}, V$$

$$= \begin{bmatrix} -0.624 & -0.449 & 0.640 \\ -0.245 & 0.890 & 0.385 \\ -0.742 & 0.083 & -0.665 \end{bmatrix}$$

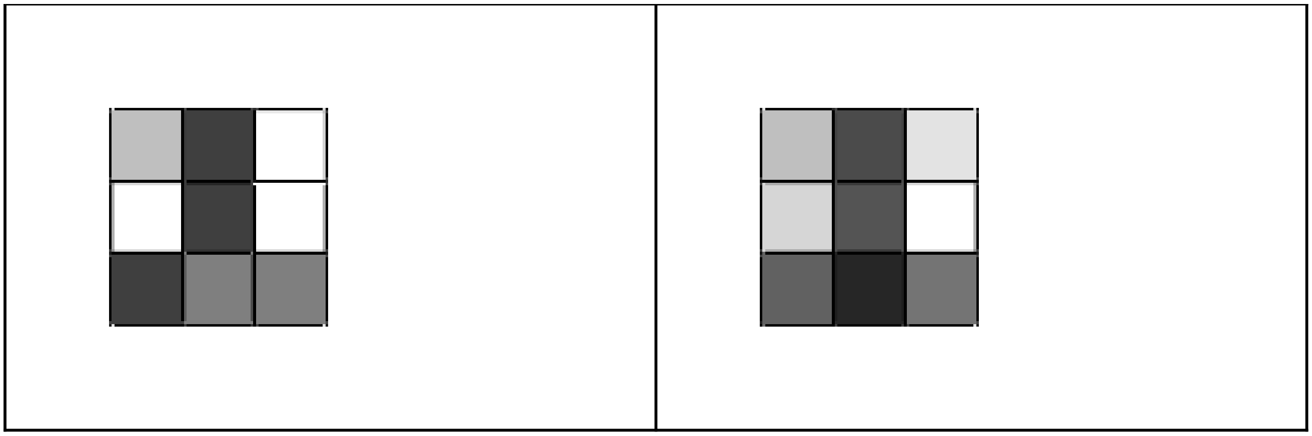
2. Using 1 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.630 \\ -0.707 \\ -0.322 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 8.075 \end{bmatrix}, V_c = \begin{bmatrix} -0.624 \\ -0.245 \\ -0.742 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 3.173 & 1.247 & 3.773 \\ 3.563 & 1.400 & 4.236 \\ 1.622 & 0.637 & 1.928 \end{bmatrix}$$



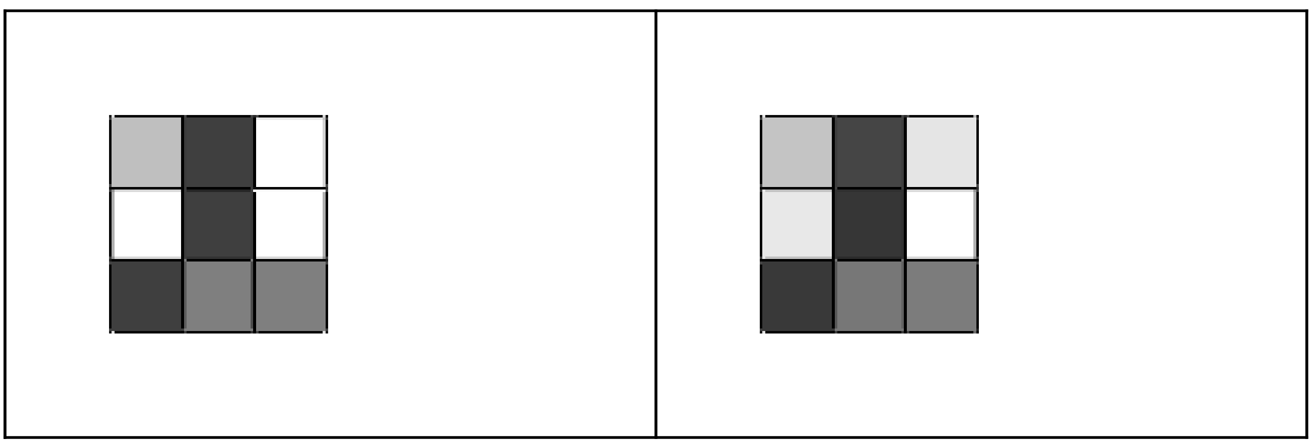
3. Using 2 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.630 & -0.077 \\ -0.707 & -0.356 \\ -0.322 & 0.931 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 8.075 & 0.000 \\ 0.000 & 1.608 \end{bmatrix}, V_c = \begin{bmatrix} -0.624 & -0.449 \\ -0.245 & 0.890 \\ -0.742 & 0.083 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 3.229 & 1.137 & 3.762 \\ 3.819 & 0.891 & 4.188 \\ 0.950 & 1.970 & 2.052 \end{bmatrix}$$



4. Using 3 singular values.

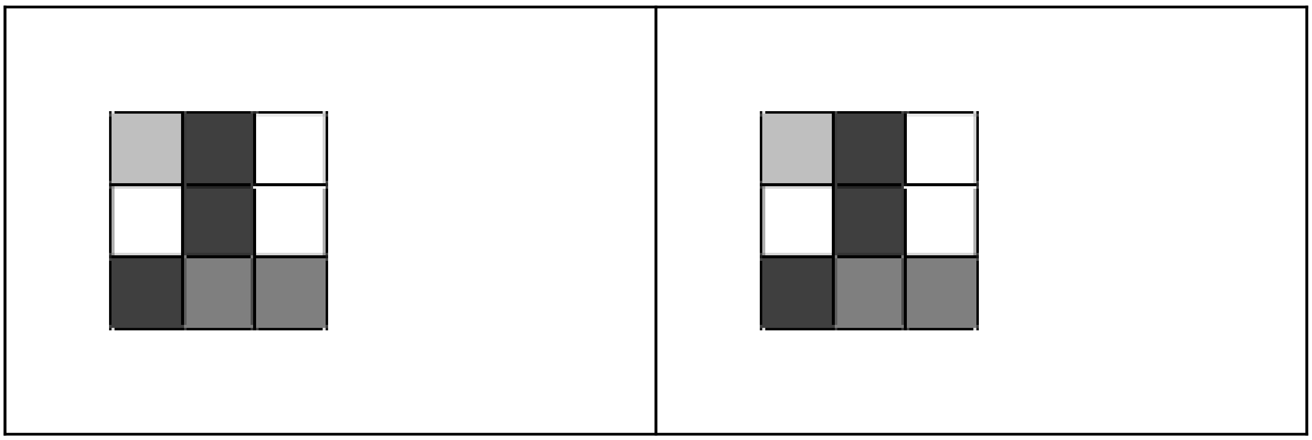
Compressing

$$U_c = \begin{bmatrix} -0.630 & -0.077 & -0.773 \\ -0.707 & -0.356 & 0.611 \\ -0.322 & 0.931 & 0.170 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 8.075 & 0.000 & 0.000 \\ 0.000 & 1.608 & 0.000 \\ 0.000 & 0.000 & 0.462 \end{bmatrix}, V_c$$

$$= \begin{bmatrix} -0.624 & -0.449 & 0.640 \\ -0.245 & 0.890 & 0.385 \\ -0.742 & 0.083 & -0.665 \end{bmatrix}$$

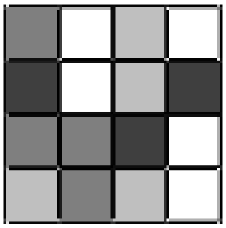
Decompressing

$$A_c = \begin{bmatrix} 3.000 & 1.000 & 4.000 \\ 4.000 & 1.000 & 4.000 \\ 1.000 & 2.000 & 2.000 \end{bmatrix}$$



Compress/Decompress the image A using SVD

$$A = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 1 & 4 & 3 & 1 \\ 2 & 2 & 1 & 4 \\ 3 & 2 & 3 & 4 \end{bmatrix}$$



1. SVD of A

$$U = \begin{bmatrix} -0.604 & -0.116 & -0.326 & -0.718 \\ -0.407 & -0.795 & 0.047 & 0.448 \\ -0.423 & 0.482 & -0.555 & 0.530 \\ -0.539 & 0.350 & 0.764 & 0.050 \end{bmatrix}, \Sigma = \begin{bmatrix} 11.068 & 0.000 & 0.000 & 0.000 \\ 0.000 & 3.258 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.344 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.289 \end{bmatrix}, V = \begin{bmatrix} -0.369 & 0.304 & 0.431 & 0.766 \\ -0.539 & -0.606 & -0.517 & 0.272 \\ -0.458 & -0.368 & 0.671 & -0.452 \\ -0.603 & 0.636 & -0.311 & -0.368 \end{bmatrix}$$

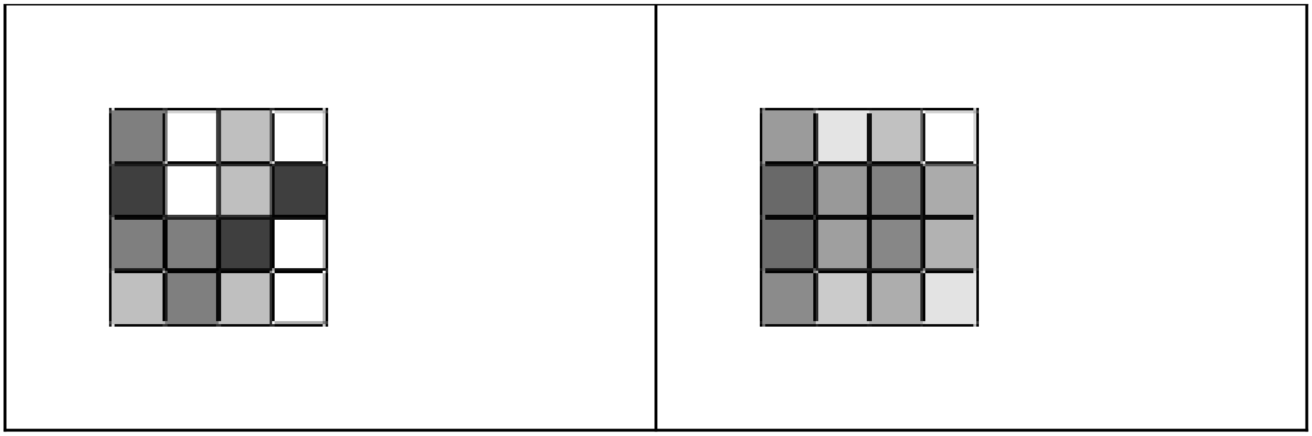
2. Using 1 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.604 \\ -0.407 \\ -0.423 \\ -0.539 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 11.068 \end{bmatrix}, V_c = \begin{bmatrix} -0.369 \\ -0.539 \\ -0.458 \\ -0.603 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 2.462 & 3.602 & 3.062 & 4.027 \\ 1.660 & 2.428 & 2.064 & 2.715 \\ 1.727 & 2.526 & 2.147 & 2.824 \\ 2.200 & 3.219 & 2.736 & 3.599 \end{bmatrix}$$



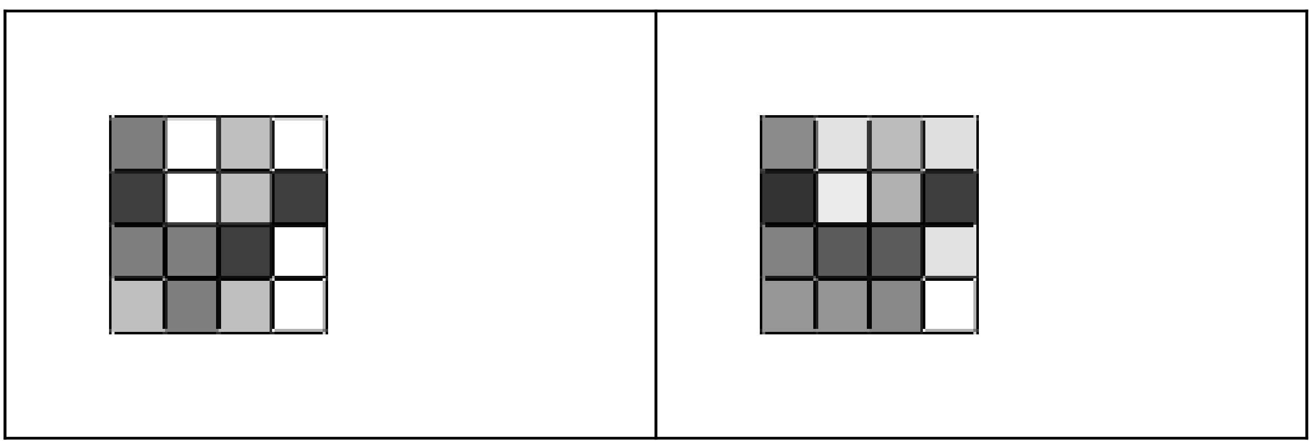
3. Using 2 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.604 & -0.116 \\ -0.407 & -0.795 \\ -0.423 & 0.482 \\ -0.539 & 0.350 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 11.068 & 0.000 \\ 0.000 & 3.258 \end{bmatrix}, V_c = \begin{bmatrix} -0.369 & 0.304 \\ -0.539 & -0.606 \\ -0.458 & -0.368 \\ -0.603 & 0.636 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 2.348 & 3.830 & 3.200 & 3.788 \\ 0.874 & 3.998 & 3.016 & 1.067 \\ 2.204 & 1.573 & 1.569 & 3.825 \\ 2.546 & 2.528 & 2.317 & 4.325 \end{bmatrix}$$



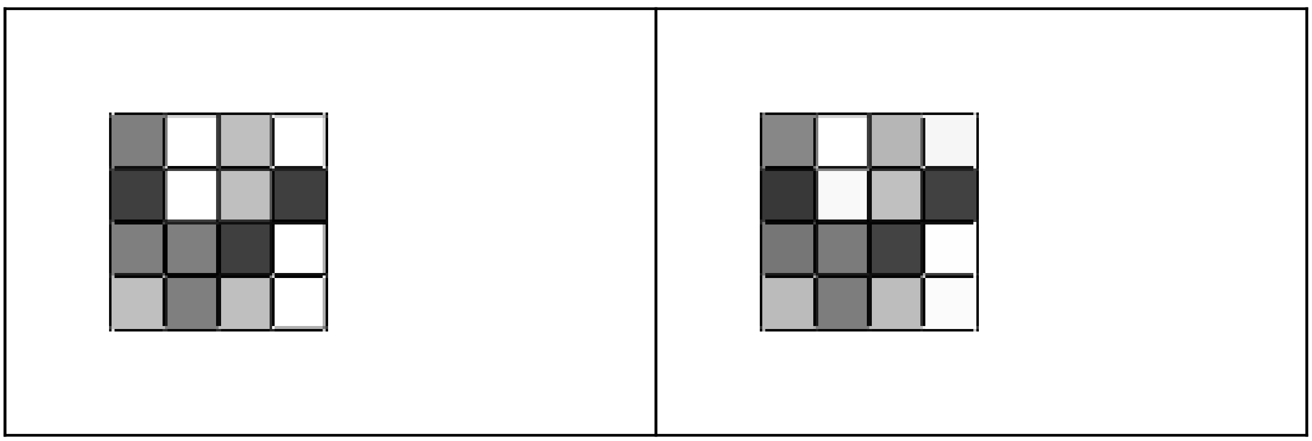
4. Using 3 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.604 & -0.116 & -0.326 \\ -0.407 & -0.795 & 0.047 \\ -0.423 & 0.482 & -0.555 \\ -0.539 & 0.350 & 0.764 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 11.068 & 0.000 & 0.000 \\ 0.000 & 3.258 & 0.000 \\ 0.000 & 0.000 & 1.344 \end{bmatrix}, V_c = \begin{bmatrix} -0.369 & 0.304 & 0.431 \\ -0.539 & -0.606 & -0.517 \\ -0.458 & -0.368 & 0.671 \\ -0.603 & 0.636 & -0.311 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 2.159 & 4.056 & 2.906 & 3.924 \\ 0.901 & 3.965 & 3.059 & 1.048 \\ 1.883 & 1.958 & 1.069 & 4.056 \\ 2.989 & 1.996 & 3.007 & 4.005 \end{bmatrix}$$



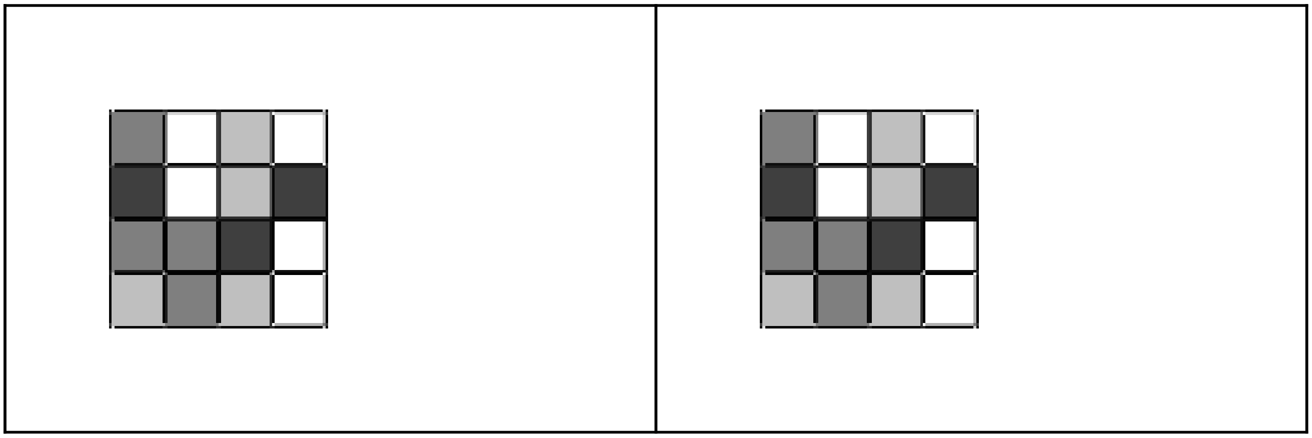
5. Using 4 singular values.

Compressing

$$U_c = \begin{bmatrix} -0.604 & -0.116 & -0.326 & -0.718 \\ -0.407 & -0.795 & 0.047 & 0.448 \\ -0.423 & 0.482 & -0.555 & 0.530 \\ -0.539 & 0.350 & 0.764 & 0.050 \end{bmatrix}, \Sigma_c = \begin{bmatrix} 11.068 & 0.000 & 0.000 & 0.000 \\ 0.000 & 3.258 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.344 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.289 \end{bmatrix}, V_c = \begin{bmatrix} -0.369 & 0.304 & 0.431 & 0.766 \\ -0.539 & -0.606 & -0.517 & 0.272 \\ -0.458 & -0.368 & 0.671 & -0.452 \\ -0.603 & 0.636 & -0.311 & -0.368 \end{bmatrix}$$

Decompressing

$$A_c = \begin{bmatrix} 2.000 & 4.000 & 3.000 & 4.000 \\ 1.000 & 4.000 & 3.000 & 1.000 \\ 2.000 & 2.000 & 1.000 & 4.000 \\ 3.000 & 2.000 & 3.000 & 4.000 \end{bmatrix}$$

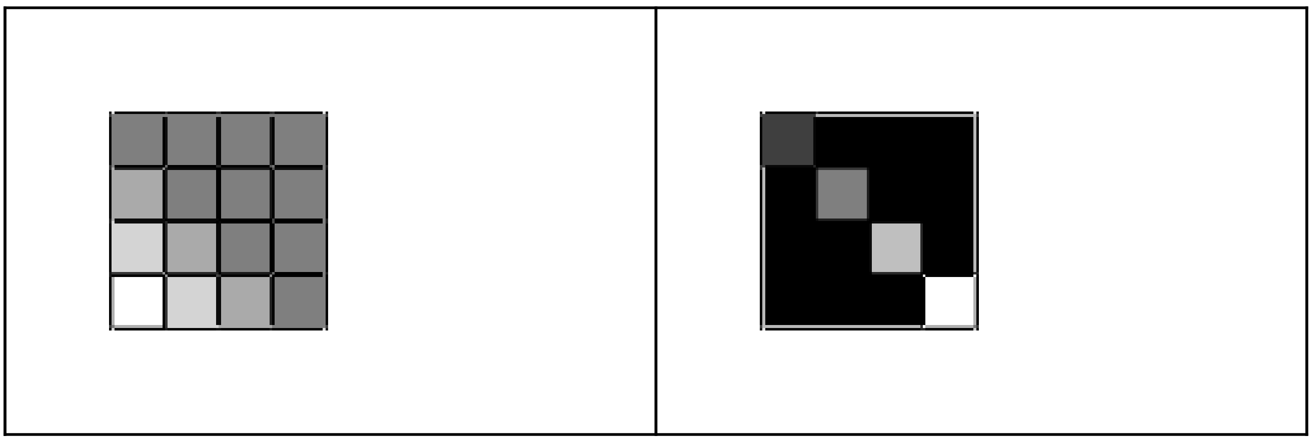


Watermarking

Images: original A and watermark W

$$A = \begin{bmatrix} 0.000 & -1.000 & -2.000 & -3.000 \\ 1.000 & 0.000 & -1.000 & -2.000 \\ 2.000 & 1.000 & 0.000 & -1.000 \\ 3.000 & 2.000 & 1.000 & 0.000 \end{bmatrix}$$

$$W = \begin{bmatrix} 2.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 4.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 6.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 8.000 \end{bmatrix}$$



1. Embed the watermark

   a. SVD of A

$$U = \begin{bmatrix} -0.837 & 0.000 & -0.006 & -0.548 \\ -0.478 & -0.267 & 0.416 & 0.726 \\ -0.120 & -0.535 & -0.815 & 0.191 \\ 0.239 & -0.802 & 0.404 & -0.369 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 4.472 & 0.000 & 0.000 & 0.000 \\ 0.000 & 4.472 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.000 & -0.837 & -0.409 & -0.364 \\ 0.267 & -0.478 & 0.817 & 0.180 \\ 0.535 & -0.120 & -0.406 & 0.731 \\ 0.802 & 0.239 & -0.002 & -0.548 \end{bmatrix}$$

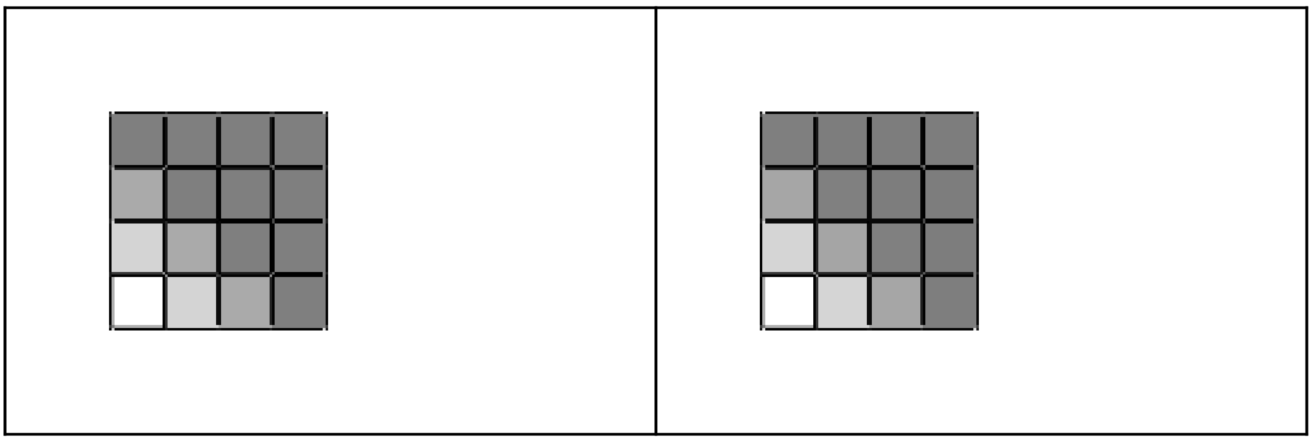
b. Embed the watermark into the singular value matrix

$k = 0.020$

$$\Sigma_w = \begin{bmatrix} 4.512 & 0.000 & 0.000 & 0.000 \\ 0.000 & 4.552 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.120 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.160 \end{bmatrix}$$

c. Construct the watermarked image

$$A_w = \begin{bmatrix} 0.032 & -1.025 & -2.082 & -2.979 \\ 0.955 & 0.067 & -0.943 & -2.084 \\ 2.065 & 0.945 & 0.065 & -1.031 \\ 3.055 & 2.062 & 0.950 & 0.025 \end{bmatrix}$$



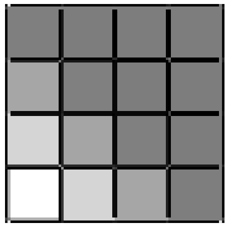
2. Bad person distorts the watermarked image

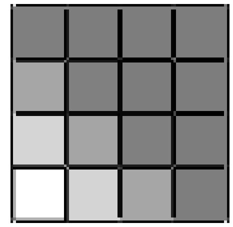
a. Filter

$$filter = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 100 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

b. Convolution

$$A_{wd} = \begin{bmatrix} 0.034 & -1.037 & -2.095 & -2.997 \\ 0.962 & 0.070 & -0.940 & -2.098 \\ 2.077 & 0.940 & 0.068 & -1.042 \\ 3.076 & 2.075 & 0.957 & 0.026 \end{bmatrix}$$



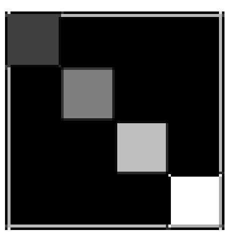


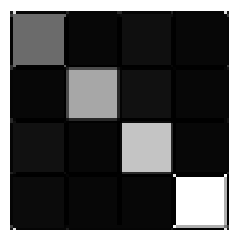
```
3. Detect the watermark
   a. Recover the watermarked singular values
```

$$\Sigma_{wd} = \begin{bmatrix} 4.540 & -0.000 & 0.005 & -0.002 \\ 0.000 & 4.580 & 0.007 & -0.003 \\ 0.007 & -0.005 & 0.128 & 0.000 \\ 0.003 & -0.002 & -0.000 & 0.167 \end{bmatrix}$$

```
   b. Recover the watermark
```

$$W_d = \begin{bmatrix} 3.388 & -0.006 & 0.263 & -0.105 \\ 0.006 & 5.401 & 0.334 & -0.163 \\ 0.349 & -0.246 & 6.397 & 0.011 \\ 0.142 & -0.123 & -0.013 & 8.370 \end{bmatrix}$$

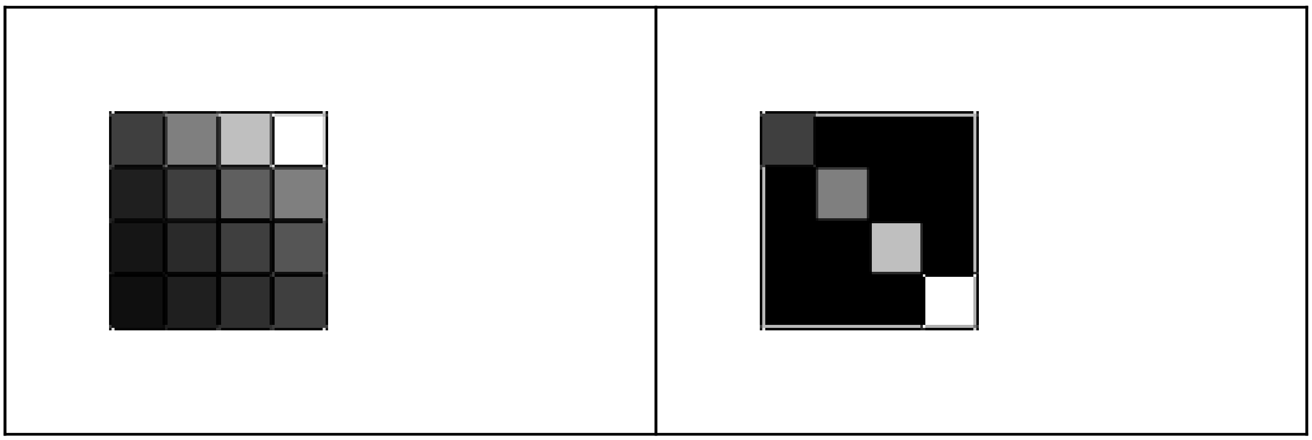




Images: original A and watermark W

$$A=\begin{bmatrix}1.000 & 2.000 & 3.000 & 4.000\\ 0.500 & 1.000 & 1.500 & 2.000\\ 0.333 & 0.667 & 1.000 & 1.333\\ 0.250 & 0.500 & 0.750 & 1.000\end{bmatrix}$$

$$W=\begin{bmatrix}2.000 & 0.000 & 0.000 & 0.000\\ 0.000 & 4.000 & 0.000 & 0.000\\ 0.000 & 0.000 & 6.000 & 0.000\\ 0.000 & 0.000 & 0.000 & 8.000\end{bmatrix}$$



1. Embed the watermark
   a. SVD of A

$$U=\begin{bmatrix}-0.838 & -0.244 & 0.488 & 0.000\\ -0.419 & -0.122 & -0.781 & -0.447\\ -0.279 & 0.960 & -0.000 & 0.000\\ -0.210 & -0.061 & -0.390 & 0.894\end{bmatrix}$$

$$\Sigma=\begin{bmatrix}6.535 & 0.000 & 0.000 & 0.000\\ 0.000 & 0.000 & 0.000 & 0.000\\ 0.000 & 0.000 & 0.000 & 0.000\\ 0.000 & 0.000 & 0.000 & 0.000\end{bmatrix}$$

$$V=\begin{bmatrix}-0.183 & 0.042 & -0.795 & 0.577\\ -0.365 & 0.503 & 0.530 & 0.577\\ -0.548 & 0.545 & -0.265 & -0.577\\ -0.730 & -0.670 & 0.132 & 0.000\end{bmatrix}$$

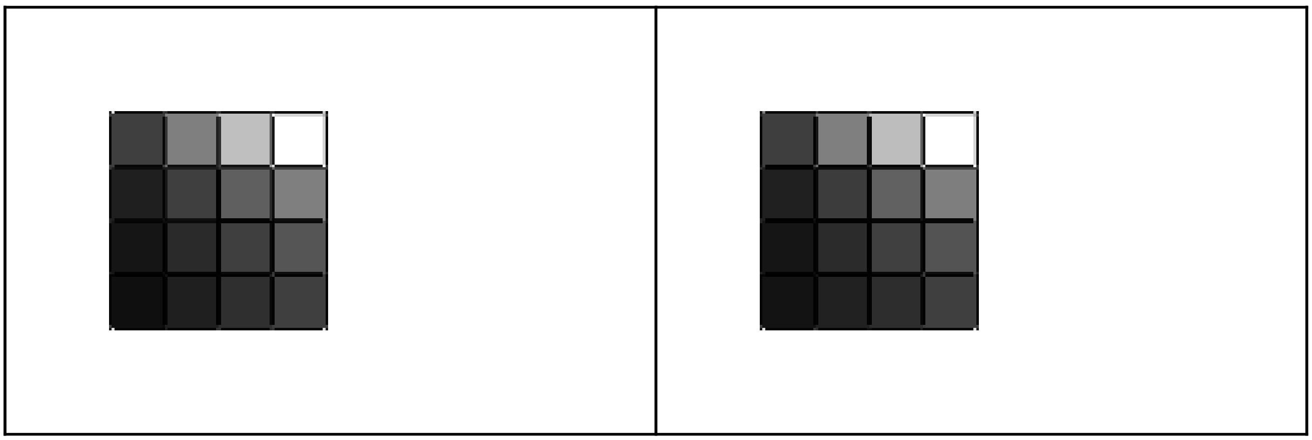
   b. Embed the watermark into the singular value matrix

$k=0.010$

$$\Sigma_w = \begin{bmatrix} 6.555 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.040 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.060 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.080 \end{bmatrix}$$

c. Construct the watermarked image

$$A_w = \begin{bmatrix} 0.979 & 2.017 & 2.996 & 4.023 \\ 0.518 & 0.955 & 1.535 & 2.003 \\ 0.336 & 0.688 & 1.024 & 1.312 \\ 0.311 & 0.529 & 0.716 & 1.002 \end{bmatrix}$$



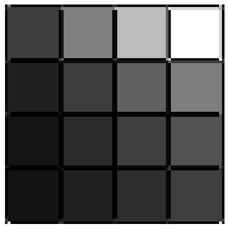
2. Bad person distorts the watermarked image

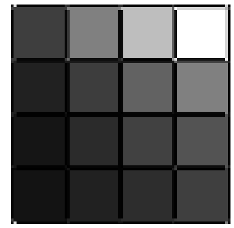
a. Filter

$$filter = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 100 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

b. Convolution

$$A_{wd} = \begin{bmatrix} 0.985 & 2.006 & 2.982 & 3.993 \\ 0.525 & 0.964 & 1.543 & 2.012 \\ 0.341 & 0.689 & 1.025 & 1.313 \\ 0.313 & 0.530 & 0.720 & 1.002 \end{bmatrix}$$





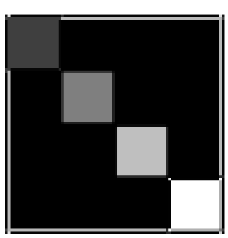
3. Detect the watermark

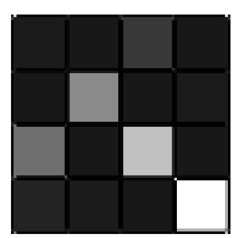
   a. Recover the watermarked singular values

$$\Sigma_{wd} = \begin{bmatrix} 6.536 & -0.008 & 0.011 & -0.007 \\ -0.008 & 0.038 & -0.001 & 0.001 \\ 0.029 & -0.000 & 0.057 & -0.001 \\ 0.004 & 0.001 & -0.001 & 0.078 \end{bmatrix}$$

   b. Recover the watermark

$$W_d = \begin{bmatrix} 0.071 & -0.845 & 1.112 & -0.669 \\ -0.843 & 3.843 & -0.065 & 0.066 \\ 2.877 & -0.031 & 5.712 & -0.109 \\ 0.386 & 0.081 & -0.123 & 7.779 \end{bmatrix}$$

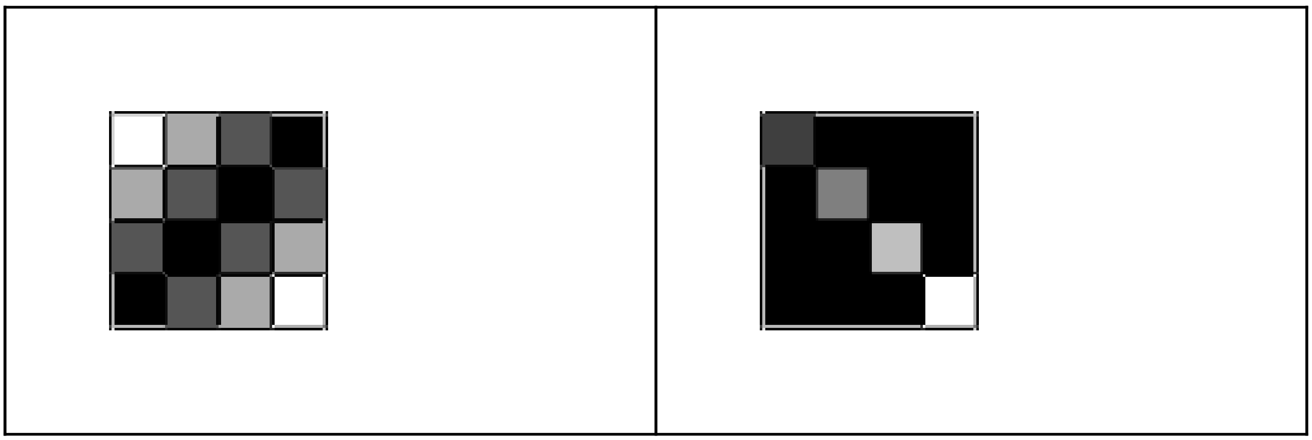




Images: original A and watermark W

$$A = \begin{bmatrix} 3.000 & 2.000 & 1.000 & 0.000 \\ 2.000 & 1.000 & 0.000 & 1.000 \\ 1.000 & 0.000 & 1.000 & 2.000 \\ 0.000 & 1.000 & 2.000 & 3.000 \end{bmatrix}$$

$$W = \begin{bmatrix} 2.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 4.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 6.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 8.000 \end{bmatrix}$$



1. Embed the watermark
   a. SVD of A

$$U = \begin{bmatrix} -0.574 & -0.653 & 0.413 & -0.271 \\ -0.413 & -0.271 & -0.574 & 0.653 \\ -0.413 & 0.271 & -0.574 & -0.653 \\ -0.574 & 0.653 & 0.413 & 0.271 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 5.162 & 0.000 & 0.000 & 0.000 \\ 0.000 & 3.414 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.162 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.586 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.574 & -0.653 & -0.413 & -0.271 \\ -0.413 & -0.271 & 0.574 & 0.653 \\ -0.413 & 0.271 & 0.574 & -0.653 \\ -0.574 & 0.653 & -0.413 & 0.271 \end{bmatrix}$$

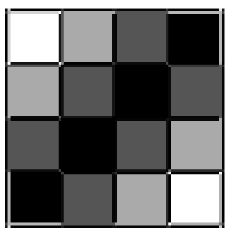
   b. Embed the watermark into the singular value matrix

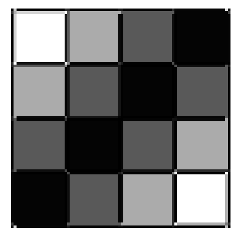
$k = 0.010$

$$\Sigma_w = \begin{bmatrix} 5.182 & 0.000 & 0.000 & 0.000 \\ 0.000 & 3.454 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.222 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.666 \end{bmatrix}$$

c. Construct the watermarked image

$$A_w = \begin{bmatrix} 3.019 & 2.012 & 1.026 & -0.027 \\ 2.012 & 1.021 & -0.053 & 1.026 \\ 1.026 & -0.053 & 1.021 & 2.012 \\ -0.027 & 1.026 & 2.012 & 3.019 \end{bmatrix}$$





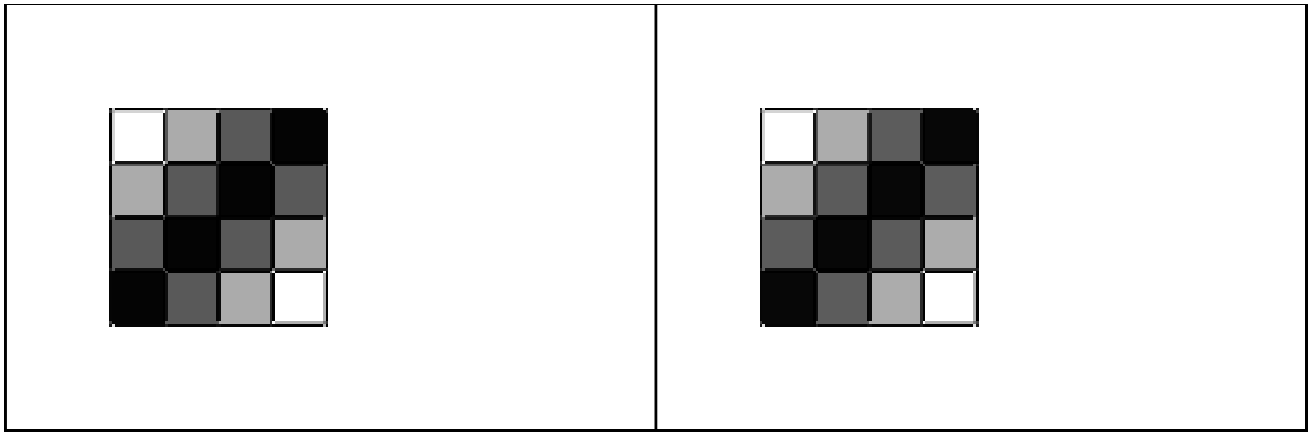
2. Bad person distorts the watermarked image

a. Filter

$$filter = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 100 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

b. Convolution

$$A_{wd} = \begin{bmatrix} 3.040 & 2.022 & 1.038 & -0.048 \\ 2.022 & 1.022 & -0.098 & 1.038 \\ 1.038 & -0.098 & 1.022 & 2.022 \\ -0.048 & 1.038 & 2.022 & 3.040 \end{bmatrix}$$



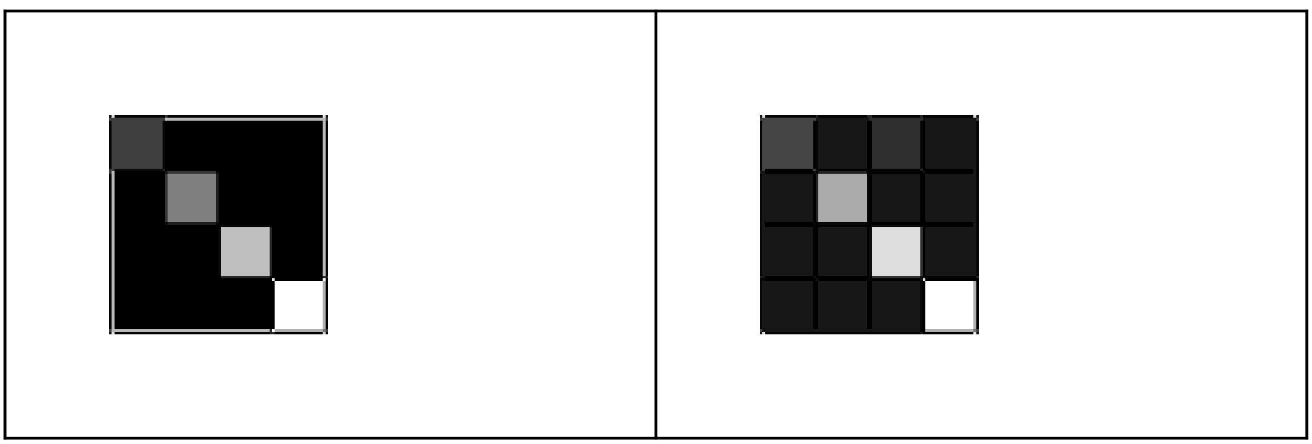
3. Detect the watermark

   a. Recover the watermarked singular values

$$\Sigma_{wd} = \begin{bmatrix} 5.188 & 0.000 & 0.013 & 0.000 \\ 0.000 & 3.496 & 0.000 & -0.000 \\ -0.013 & 0.000 & 1.272 & 0.000 \\ 0.000 & -0.000 & 0.000 & 0.713 \end{bmatrix}$$

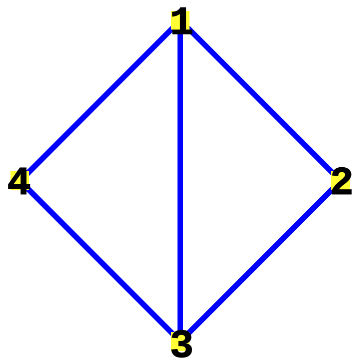
   b. Recover the watermark

$$W_d = \begin{bmatrix} 2.534 & 0.000 & 1.313 & 0.000 \\ 0.000 & 8.142 & 0.000 & -0.032 \\ -1.313 & 0.000 & 10.940 & 0.000 \\ 0.000 & -0.032 & 0.000 & 12.722 \end{bmatrix}$$



Graph partition: Facebook Community Identification

Partition the graph into two subgraphs with minimum extra-cut size:



1. Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

2. Modulation Matrix

$$B = \begin{bmatrix} -\frac{9}{10} & \frac{2}{5} & \frac{1}{10} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{1}{10} & \frac{2}{5} & -\frac{9}{10} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

3. Eigen decomposition of B

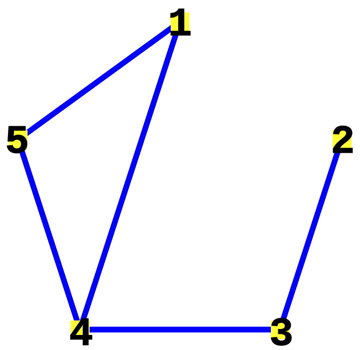
$$\Lambda = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -1.600 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.500 & -0.057 & -0.707 & -0.500 \\ -0.500 & -0.760 & 0.000 & 0.500 \\ -0.500 & -0.057 & 0.707 & -0.500 \\ -0.500 & 0.645 & 0.000 & 0.500 \end{bmatrix}$$

`4. Partition`

$G1 = [\ ]$, $G2 = [1, 2, 3, 4]$

Partition the graph into two subgraphs with minimum extra-cut size:



1. Adjacency Matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Modulation Matrix

$$B = \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} & \frac{3}{5} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{4}{5} & -\frac{3}{10} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{10} & \frac{2}{5} & -\frac{9}{10} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

3. Eigen decomposition of B

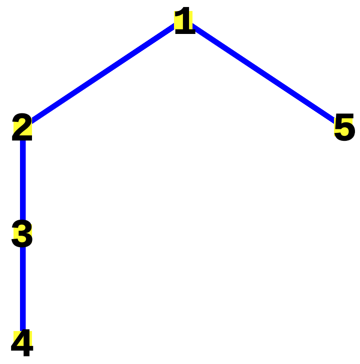
$$\Lambda = \begin{bmatrix} 1.048 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.569 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & -1.679 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.450 & 0.447 & 0.217 & 0.707 & 0.225 \\ 0.555 & 0.447 & 0.602 & 0.000 & -0.360 \\ 0.510 & 0.447 & -0.460 & 0.000 & 0.573 \\ -0.166 & 0.447 & -0.576 & 0.000 & -0.664 \\ -0.450 & 0.447 & 0.217 & -0.707 & 0.225 \end{bmatrix}$$

4. Partition

$G1 = [2, 3], G2 = [1, 4, 5]$

Partition the graph into two subgraphs with minimum extra-cut size:



1. Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Modulation Matrix

$$B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

3. Eigen decomposition of B

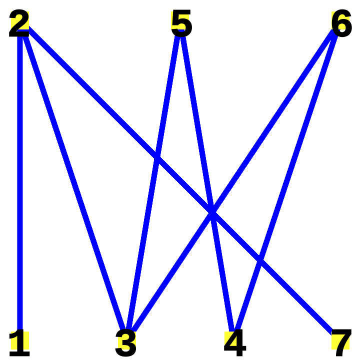
$$\Lambda = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & -1.750 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.500 & 0.363 & 0.463 & -0.500 & -0.535 \\ 0.000 & 0.126 & 0.617 & 0.000 & 0.535 \\ -0.500 & 0.363 & 0.463 & 0.500 & -0.535 \\ -0.500 & 0.600 & 0.309 & -0.500 & 0.267 \\ 0.500 & 0.600 & 0.309 & 0.500 & 0.267 \end{bmatrix}$$

4. Partition

$G1 = [1, 2, 5]$, $G2 = [3, 4]$

Partition the graph into two subgraphs with minimum extra-cut size:



1. Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Modulation Matrix

$$B = \begin{bmatrix} -\frac{1}{14} & \frac{11}{14} & -\frac{3}{14} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{14} \\ \frac{11}{14} & -\frac{9}{14} & \frac{5}{14} & -\frac{3}{7} & -\frac{3}{7} & -\frac{3}{7} & \frac{11}{14} \\ -\frac{3}{14} & \frac{5}{14} & -\frac{9}{14} & -\frac{3}{7} & \frac{4}{7} & \frac{4}{7} & -\frac{3}{14} \\ -\frac{1}{7} & -\frac{3}{7} & -\frac{3}{7} & -\frac{2}{7} & \frac{5}{7} & \frac{5}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{3}{7} & \frac{4}{7} & \frac{5}{7} & -\frac{2}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{3}{7} & \frac{4}{7} & \frac{5}{7} & -\frac{2}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{11}{14} & -\frac{3}{14} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{14} \end{bmatrix}$$

3. Eigen decomposition of B

$$\Lambda = \begin{bmatrix} 1.428 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -1.475 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -2.238 \end{bmatrix}$$

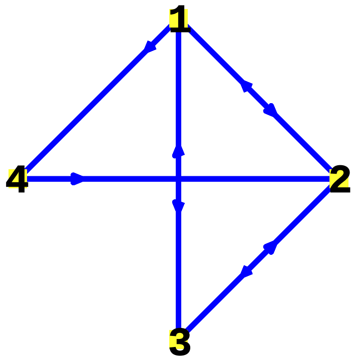
$$V = \begin{bmatrix} 0.372 & 0.000 & 0.835 & -0.799 & 0.202 & 0.363 & -0.182 \\ 0.507 & 0.000 & 0.145 & -0.053 & -0.310 & -0.640 & 0.427 \\ -0.088 & 0.000 & -0.036 & 0.273 & -0.491 & -0.095 & -0.535 \\ -0.460 & 0.000 & 0.327 & -0.380 & -0.129 & -0.458 & -0.353 \\ -0.351 & -0.707 & 0.145 & -0.053 & -0.310 & 0.233 & 0.413 \\ -0.351 & 0.707 & 0.145 & -0.053 & -0.310 & 0.233 & 0.413 \\ 0.372 & 0.000 & -0.363 & 0.366 & -0.642 & 0.363 & -0.182 \end{bmatrix}$$

4. Partition

$G1 = [1, 2, 7]$, $G2 = [3, 4, 5, 6]$

Google Page Ranking

Rank the web pages with the following links:



1. Construct the transition matrix

$$T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

2. Initialize the probability vector

$$P_0 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

3. Repeatedly multiply by T until the change is less than 0.010

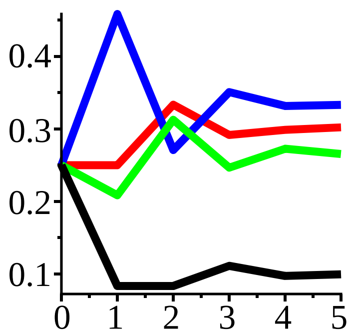
$$P_1 = \begin{bmatrix} 0.250 \\ 0.458 \\ 0.208 \\ 0.083 \end{bmatrix}, P_1 - P_0 = \begin{bmatrix} 0.000 \\ 0.208 \\ -0.042 \\ -0.167 \end{bmatrix}, maxnorm(P_1 - P_0) = 0.208$$

$$P_2 = \begin{bmatrix} 0.333 \\ 0.271 \\ 0.312 \\ 0.083 \end{bmatrix}, P_2 - P_1 = \begin{bmatrix} 0.083 \\ -0.188 \\ 0.104 \\ 0.000 \end{bmatrix}, maxnorm(P_2 - P_1) = 0.188$$

$$P_3 = \begin{bmatrix} 0.292 \\ 0.351 \\ 0.247 \\ 0.111 \end{bmatrix}, P_3 - P_2 = \begin{bmatrix} -0.042 \\ 0.080 \\ -0.066 \\ 0.028 \end{bmatrix}, maxnorm(P_3 - P_2) = 0.080$$

$$P_4 = \begin{bmatrix} 0.299 \\ 0.332 \\ 0.273 \\ 0.097 \end{bmatrix}, P_4 - P_3 = \begin{bmatrix} 0.007 \\ -0.019 \\ 0.026 \\ -0.014 \end{bmatrix}, maxnorm(P_4 - P_3) = 0.026$$

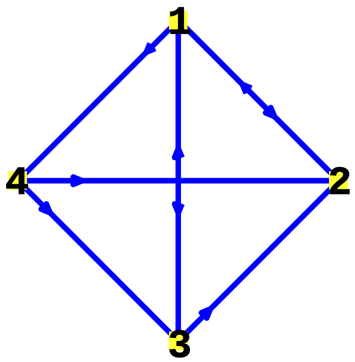
$$P_5 = \begin{bmatrix} 0.302 \\ 0.333 \\ 0.265 \\ 0.100 \end{bmatrix}, P_5 - P_4 = \begin{bmatrix} 0.003 \\ 0.001 \\ -0.007 \\ 0.002 \end{bmatrix}, maxnorm(P_5 - P_4) = 0.007$$



4. Rank the web pages

2
1
3
4

Rank the web pages with the following links:



1. Construct the transition matrix

$$T = \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

2. Initialize the probability vector

$$P_0 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

3. Repeatedly multiply by T until the change is less than 0.010

$$P_1 = \begin{bmatrix} 0.375 \\ 0.333 \\ 0.208 \\ 0.083 \end{bmatrix}, P_1 - P_0 = \begin{bmatrix} 0.125 \\ 0.083 \\ -0.042 \\ -0.167 \end{bmatrix}, maxnorm(P_1 - P_0) = 0.167$$

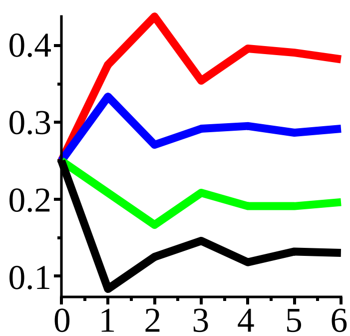
$$P_2 = \begin{bmatrix} 0.437 \\ 0.271 \\ 0.167 \\ 0.125 \end{bmatrix}, P_2 - P_1 = \begin{bmatrix} 0.062 \\ -0.063 \\ -0.042 \\ 0.042 \end{bmatrix}, maxnorm(P_2 - P_1) = 0.063$$

$$P_3 = \begin{bmatrix} 0.354 \\ 0.292 \\ 0.208 \\ 0.146 \end{bmatrix}, P_3 - P_2 = \begin{bmatrix} -0.083 \\ 0.021 \\ 0.042 \\ 0.021 \end{bmatrix}, maxnorm(P_3 - P_2) = 0.083$$

$$P_4 = \begin{bmatrix} 0.396 \\ 0.295 \\ 0.191 \\ 0.118 \end{bmatrix}, P_4 - P_3 = \begin{bmatrix} 0.042 \\ 0.003 \\ -0.017 \\ -0.028 \end{bmatrix}, maxnorm(P_4 - P_3) = 0.042$$

$$P_5 = \begin{bmatrix} 0.391 \\ 0.286 \\ 0.191 \\ 0.132 \end{bmatrix}, P_5 - P_4 = \begin{bmatrix} -0.005 \\ -0.009 \\ 0.000 \\ 0.014 \end{bmatrix}, maxnorm(P_5 - P_4) = 0.014$$

$$P_6 = \begin{bmatrix} 0.382 \\ 0.292 \\ 0.196 \\ 0.130 \end{bmatrix}, P_6 - P_5 = \begin{bmatrix} -0.009 \\ 0.005 \\ 0.005 \\ -0.002 \end{bmatrix}, maxnorm(P_6 - P_5) = 0.009$$



4. Rank the web pages

1
2
3
4