MA 303

Linear Difference Equation

1 Homogeneous: 1st-order 1-variable

1. $y_{k+1} + 2y_k = 0$
   $y_0 = 6$

   (a) Find the general solution
   $\lambda + 2 = 0$
   $\lambda = -2$
   $y_k = C\,(-2)^k$

   (b) Find the particular solution
   $6 = C\,(-2)^0$
   $C = 6$
   $y_k = 6 \cdot (-2)^k$

2. $y_{k+1} - 3y_k = 0$
   $y_0 = 2$

   (a) Find the general solution
   $\lambda - 3 = 0$
   $\lambda = 3$
   $y_k = C3^k$

   (b) Find the particular solution
   $2 = C3^0$
   $C = 2$
   $y_k = 2 \cdot 3^k$

2 Homogeneous: 2nd-order 1-variable

1. $y_{k+2} - 7y_{k+1} + 12y_k = 0$
   $y_0 = 3$
   $y_1 = 11$

   (a) Find the general solution
   $\lambda^2 - 7\lambda + 12 = 0$
   $(\lambda - 3)(\lambda - 4) = 0$
   $\lambda = 3, 4$
   $y_k = C_1 3^k + C_2 4^k$

   (b) Find the particular solution
   $3 = C_1 3^0 + C_2 4^0$
   $11 = C_1 3^1 + C_2 4^1$
   $3 = C_1 + C_2$
   $11 = 3C_1 + 4C_2$
   $C_1 = 1$
   $C_2 = 2$
   $y_k = 3^k + 2 \cdot 4^k$

2. $y_{k+2} + 5y_{k+1} + 6y_k = 0$
   $y_0 = 3$
   $y_1 = 0$

   (a) Find the general solution
   $\lambda^2 + 5\lambda + 6 = 0$
   $(\lambda + 3)(\lambda + 2) = 0$
   $\lambda = -3, -2$
   $y_k = C_1 (-3)^k + C_2 (-2)^k$

   (b) Find the particular solution
   $3 = C_1 (-3)^0 + C_2 (-2)^0$
   $0 = C_1 (-3)^1 + C_2 (-2)^1$
   $3 = C_1 + C_2$
   $0 = 3C_1 + 4C_2$
   $C_1 = -6$
   $C_2 = 9$
   $y_k = -6 \cdot (-3)^k + 9 \cdot (-2)^k$

3. $y_{k+2} - 6y_{k+1} + 9y_k = 0$

   $y_0 = 2$

   $y_1 = 3$

   (a) Find the general solution

   $\lambda^2 - 6\lambda + 9 = 0$

   $(\lambda - 3)(\lambda - 3) = 0$

   $\lambda = 3, 3$

   $y_k = C_1 3^k + C_2 k 3^k$

   (b) Find the particular solution

   $2 = C_1 3^0 + C_2 \cdot 0 \cdot 3^0$

   $3 = C_1 3^1 + C_2 \cdot 1 \cdot 3^1$

   $2 = C_1$

   $3 = 3C_1 + 3C_2$

   $C_1 = 2$

   $C_2 = -1$

   $y_k = 2 \cdot 3^k - k3^k$

4. $y_{k+2} + 4y_{k+1} + 4y_k = 0$

   $y_0 = 3$

   $y_1 = 6$

   (a) Find the general solution

   $\lambda^2 + 4\lambda + 4 = 0$

   $(\lambda + 2)(\lambda + 2) = 0$

   $\lambda = -2, -2$

   $y_k = C_1 (-2)^k + C_2 k (-2)^k$

   (b) Find the particular solution

   $3 = C_1 (-2)^0 + C_2 \cdot 0 \cdot (-2)^0$

   $6 = C_1 (-2)^1 + C_2 \cdot 1 \cdot (-2)^1$

   $3 = C_1$

   $6 = -2C_1 - 2C_2$

   $C_1 = 3$

   $C_2 = -6$

   $y_k = 3 \cdot (-2)^k - 6 \cdot k (-2)^k$

5. $y_{k+2} - 4y_{k+1} + 8y_k = 0$

   $y_0 = 3$

   $y_1 = 4$

   (a) Find the general solution

   $\lambda^2 - 4\lambda + 8 = 0$

   $\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 8}}{2} = 2 \pm 2i$

   $r = \sqrt{2^2 + 2^2} = \sqrt{8}$

   $\theta = \arctan \frac{2}{2} = \frac{\pi}{4}$

   $y_k = \left(\sqrt{8}\right)^k \left(C_1 \cos\left(k\frac{\pi}{4}\right) + C_2 \sin\left(k\frac{\pi}{4}\right)\right)$

   (b) Find the particular solution

   $3 = \left(\sqrt{8}\right)^0 \left(C_1 \cos\left(0 \cdot \frac{\pi}{4}\right) + C_2 \sin\left(0 \cdot \frac{\pi}{4}\right)\right)$

   $4 = \left(\sqrt{8}\right)^1 \left(C_1 \cos\left(1 \cdot \frac{\pi}{4}\right) + C_2 \sin\left(1 \cdot \frac{\pi}{4}\right)\right)$

   $3 = C_1$

   $4 = 2C_1 + 2C_2$

   $C_1 = 3$

   $C_2 = -1$

   $y_k = \left(\sqrt{8}\right)^k \left(3\cos\left(k\frac{\pi}{4}\right) - \sin\left(k\frac{\pi}{4}\right)\right)$

6. $y_{k+2} - 2y_{k+1} + 4y_k = 0$

   $y_0 = 3$

   $y_1 = 2$

   (a) Find the general solution

   $\lambda^2 - 2\lambda + 4 = 0$

   $\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2} = 1 \pm \sqrt{3}i$

   $r = \sqrt{1^2 + \sqrt{3}^2} = 2$

   $\theta = \arctan \frac{\sqrt{3}}{1} = \frac{\pi}{3}$

   $y_k = 2^k \left(C_1 \cos\left(k\frac{\pi}{3}\right) + C_2 \sin\left(k\frac{\pi}{3}\right)\right)$

   (b) Find the particular solution

   $3 = 2^0 \left(C_1 \cos\left(0 \cdot \frac{\pi}{3}\right) + C_2 \sin\left(0 \cdot \frac{\pi}{3}\right)\right)$

   $2 = 2^1 \left(C_1 \cos\left(1 \cdot \frac{\pi}{3}\right) + C_2 \sin\left(1 \cdot \frac{\pi}{3}\right)\right)$

   $3 = C_1$

   $2 = C_1 + \sqrt{3}C_2$

   $C_1 = 3$

   $C_2 = -\frac{1}{\sqrt{3}}$

   $y_k = 2^k \left(3\cos\left(k\frac{\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\left(k\frac{\pi}{3}\right)\right)$

3 Non-homogeneous: 1st-order 1-variable

1. $y_{k+1} - 3y_k = -4 \cdot 5^k$
   $y_0 = 4$

   (a) Find the general solution of the homogeneous equation
   $\lambda - 3 = 0$
   $\lambda = 3$
   $y_k = C3^k$

   (b) Find a particular solution of the non-homogeneous equation
   Try $y_k = D5^k$
   $D5^{k+1} - 3D5^k = -4 \cdot 5^k$
   $5D - 3D = -4$
   $2D = -4$
   $D = -2$
   $y_k = -2 \cdot 5^k$

   (c) Find the general solution of the non-homogeneous equation
   $y_k = C3^k - 2 \cdot 5^k$

   (d) Find the particular solution of the non-homogeneous equation
   $4 = C3^0 - 2 \cdot 5^0$
   $4 = C - 2$
   $C = 6$
   $y_k = 6 \cdot 3^k - 2 \cdot 5^k$

2. $y_{k+1} - 4y_k = -5 \cdot 3^k$
   $y(0) = 3$

   (a) Find the general solution of the homogeneous equation
   $\lambda - 4 = 0$
   $\lambda = 4$
   $y_k = C4^k$

   (b) Find a particular solution of the non-homogeneous equation
   Try $y_k = D3^k$
   $D3^{k+1} - 4D3^k = -5 \cdot 3^k$
   $3D - 4D = -5$
   $D = 5$
   $y_k = 5 \cdot 3^k$

   (c) Find the general solution of the non-homogeneous equation
   $y_k = C4^k + 5 \cdot 3^k$

   (d) Find the particular solution of the non-homogeneous equation
   $3 = C4^0 + 5 \cdot 3^0$
   $3 = C + 5$
   $C = -2$
   $y_k = -2 \cdot 4^k + 5 \cdot 3^k$

3. $y_{k+1} - 4y_k = -7 \cdot 4^k$
   $y(0) = 7$

   (a) Find the general solution of the homogeneous equation
   $\lambda - 4 = 0$
   $\lambda = 4$
   $y_k = C4^k$

   (b) Find a particular solution of the non-homogeneous equation
   Try $y_k = Dk4^k$
   $D(k+1)4^{k+1} - 4Dk4^k = -7 \cdot 4^k$
   $D(k+1)4 - 4kD = -7$
   $4D = -7$
   $D = -\frac{7}{4}$
   $y_k = -\frac{7}{4} \cdot k4^k$

   (c) Find the general solution of the non-homogeneous equation
   $y_k = C4^k - \frac{7}{4} \cdot k4^k$

   (d) Find the particular solution of the non-homogeneous equation
   $7 = C4^0 - \frac{7}{4} \cdot 0 \cdot 4^0$
   $C = 7$
   $y_k = 7 \cdot 4^k - \frac{7}{4} \cdot k4^k$

4. $y_{k+1} + 3y_k = 2 \cdot (-3)^k$
   $y(0) = 2$

   (a) Find the general solution of the homogeneous equation
   $\lambda + 3 = 0$
   $\lambda = -3$
   $y_k = C(-3)^k$

   (b) Find a particular solution of the non-homogeneous equation
   Try $y_k = Dk(-3)^k$
   $D(k+1)(-3)^{k+1} + 3Dk(-3)^k = 2 \cdot (-3)^k$
   $D(k+1)(-3) + 3Dk = 2$
   $-3D = 2$
   $D = -\frac{2}{3}$
   $y_k = -\frac{2}{3} \cdot k(-3)^k$

   (c) Find the general solution of the non-homogeneous equation
   $y_k = C(-3)^k - \frac{2}{3} \cdot k(-3)^k$

   (d) Find the particular solution of the non-homogeneous equation
   $2 = C(-3)^0 - \frac{2}{3} \cdot 0 \cdot (-3)^0$
   $C = 2$
   $y_k = 2 \cdot (-3)^k - \frac{2}{3} \cdot k(-3)^k$

4 Non-homogeneous: 2nd-order 1-variable

1. $y_{k+2} - 7y_{k+1} + 10y_k = 7 \cdot 4^k$
   $y_0 = 4$
   $y_1 = 13$

   (a) Find the general solution of the homogeneous equation
   $\lambda^2 - 7\lambda + 10\lambda = 0$
   $(\lambda - 2)(\lambda - 5) = 0$
   $\lambda = 2, 5$
   $y_k = C_1 2^k + C_2 5^k$

   (b) Find any solution of the non-homogeneous equation
   Try $y_k = D4^k$
   $D4^{k+2} - 7D4^{k+1} + 10D4^k = 7 \cdot 4^k$
   $D4^2 - 7D4 + 10D = 7$
   $-2D = 7$
   $D = -\frac{7}{2}$
   $y_k = -\frac{7}{2} \cdot 4^k$

   (c) Find the general solution of the non-homogeneous equation
   $y_k = C_1 2^k + C_2 5^k - \frac{7}{2} 4^k$

   (d) Find the particular solution of the non-homogeneous equation
   $4 = C_1 2^0 + C_2 5^0 - \frac{7}{2} \cdot 4^0$
   $13 = C_1 2^1 + C_2 5^1 - \frac{7}{2} \cdot 4^1$
   $4 = C_1 + C_2 - \frac{7}{2}$
   $13 = 2C_1 + 5C_2 - \frac{7}{2} \cdot 4$
   $4 + \frac{7}{2} = C_1 + C_2$
   $13 + \frac{7}{2} \cdot 4 = 2C_1 + 5C_2$
   $C_1 = \frac{7}{2}$
   $C_2 = 4$
   $y_k = \frac{7}{2} \cdot 2^k + 4 \cdot 5^k - \frac{7}{2} \cdot 4^k$

2. $y_{k+2} - 5y_{k+1} + 6y_k = 6 \cdot 4^k$

   $y_0 = 0$

   $y_1 = 1$

   (a) Find the general solution of the homogeneous equation

   $\lambda^2 - 5\lambda + 6\lambda = 0$

   $(\lambda - 2)(\lambda - 3) = 0$

   $\lambda = 2, 3$

   $y_k = C_1 2^k + C_2 3^k$

   (b) Find any solution of the non-homogeneous equation

   Try $y_k = D4^k$

   $D4^{k+2} - 5D4^{k+1} + 6D4^k = 6 \cdot 4^k$

   $D4^2 - 5D4 + 6D = 6$

   $2D = 6$

   $D = 3$

   $y_k = 3 \cdot 4^k$

   (c) Find the general solution of the non-homogeneous equation

   $y_k = C_1 2^k + C_2 3^k + 3 \cdot 4^k$

   (d) Find the particular solution of the non-homogeneous equation

   $0 = C_1 2^0 + C_2 3^0 + 3 \cdot 4^0$

   $1 = C_1 2^1 + C_2 3^1 + 3 \cdot 4^1$

   $0 = C_1 + C_2 + 3$

   $1 = 2C_1 + 3C_2 + 12$

   $-3 = C_1 + C_2$

   $-11 = 2C_1 + 3C_2$

   $C_1 = 2$

   $C_2 = -5$

   $y_k = 2 \cdot 2^k - 5 \cdot 3^k + 3 \cdot 4^k$

3. $y_{k+2} - 9y_{k+1} + 14y_k = 30 \cdot 2^k$

   $y_0 = 1$

   $y_1 = 6$

   (a) Find the general solution of the homogeneous equation

   $\lambda^2 - 9\lambda + 14\lambda = 0$

   $(\lambda - 2)(\lambda - 7) = 0$

   $\lambda = 2, 7$

   $y_k = C_1 2^k + C_2 7^k$

   (b) Find any solution of the non-homogeneous equation

   Try $y_k = Dk2^k$

   $D(k+2)2^{k+2} - 9D(k+1)2^{k+1} + 14Dk2^k = 30 \cdot 2^k$

   $D(k+2)2^2 - 9D(k+1)2 + 14Dk = 30$

   $D\left((k+2)2^2 - 9(k+1)2 + 14k\right) = 30$

   $D(-10) = 30$

   $D = -3$

   $y_k = -3k2^k$

   (c) Find the general solution of the non-homogeneous equation

   $y_k = C_1 2^k + C_2 7^k - 3k2^k$

   (d) Find the particular solution of the non-homogeneous equation

   $1 = C_1 2^0 + C_2 7^0 - 3 \cdot 0 \cdot 2^0$

   $6 = C_1 2^1 + C_2 7^1 - 3 \cdot 1 \cdot 2^1$

   $1 = C_1 + C_2$

   $6 = 2C_1 + 7C_2 - 6$

   $1 = C_1 + C_2$

   $12 = 2C_1 + 7C_2$

   $C_1 = -1$

   $C_2 = 2$

   $y_k = -2^k + 2 \cdot 7^k - 3 \cdot k2^k$

4. $y_{k+2} - 5y_{k+1} + 6y_k = 6 \cdot 3^k$

   $y_0 = 1$

   $y_1 = 2$

   (a) Find the general solution of the homogeneous equation

   $\lambda^2 - 5\lambda + 6\lambda = 0$

   $(\lambda - 2)(\lambda - 3) = 0$

   $\lambda = 2, 3$

   $y_k = C_1 2^k + C_2 3^k$

   (b) Find any solution of the non-homogeneous equation

   Try $y_k = Dk3^k$

   $D(k+2)3^{k+2} - 5D(k+1)3^{k+1} + 6Dk3^k = 6 \cdot 3^k$

   $D(k+2)3^2 - 5D(k+1)3 + 6Dk = 6$

   $3D = 6$

   $D = 2$

   $y_k = 2 \cdot k3^k$

   (c) Find the general solution of the non-homogeneous equation

   $y_k = C_1 2^k + C_2 3^k + 2 \cdot k3^k$

   (d) Find the particular solution of the non-homogeneous equation

   $1 = C_1 2^0 + C_2 3^0 + 2 \cdot 0 \cdot 3^0$

   $2 = C_1 2^1 + C_2 3^1 + 2 \cdot 1 \cdot 3^1$

   $1 = C_1 + C_2$

   $2 = 2C_1 + 3C_2 + 6$

   $1 = C_1 + C_2$

   $-4 = 2C_1 + 3C_2$

   $C_1 = 7$

   $C_2 = -6$

   $y_k = 7 \cdot 2^k - 6 \cdot 3^k + 2 \cdot k3^k$