

MA 242 Test 4

Multivariate Vector Functions ($\mathbb{R}^n \rightarrow \mathbb{R}^n$)

Last Name : _____ First Name: _____ Student ID: _____ Seat Code: _____

1. [1.0 points] Write the precise statement of the following theorems:

- Curve Parameterization

$$\int_C f \cdot dr = \int_{t_1}^{t_2} g dt$$

where

$$g = f \cdot \frac{dr}{dt}$$

$[t_1, t_2]$ is the domain for the parameter t .

- Surface Parameterization

$$\int_S f \cdot ds = \int_D g dA$$

where

$$g = f \cdot (r_u \times r_v), \text{ RHR}$$

D is the domain for the parameters (u, v) .

- Fundamental theorem of curve (line) integral

$$\int_C f \cdot dr = [g]_{r_1}^{r_2}$$

where

$$\nabla g = f$$

$$\{r_1, r_2\} = \partial C.$$

- Stoke's theorem (curve integral \rightarrow surface integral)

$$\int_C f \cdot dr = \int_S g \cdot ds$$

where

$$g = \nabla \times f$$

$$C = \partial S, \text{ RHR.}$$

- Stoke's theorem (surface integral \rightarrow curve integral)

$$\int_S f \cdot ds = \int_C g \cdot dr$$

where

$$\nabla \times g = f$$

$$C = \partial S, \text{ RHR.}$$

- Gauss' theorem

$$\int_S f \cdot ds = \int_D g dV$$

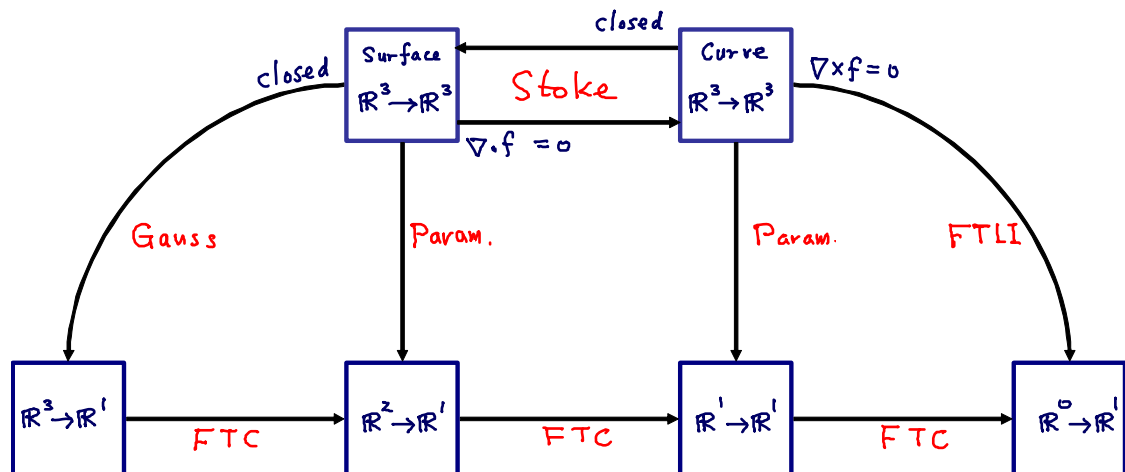
where

$$g = \nabla \cdot f$$

$$S = \partial D, \text{ outward.}$$

2. [1.0 points] Draw the following diagrams:

- Roadmap



- Driving Instruction for Curve Integral

- C is open and $\nabla \times f \neq 0$: Param
- C is open and $\nabla \times f = 0$: FTLI
- C is closed and $\nabla \times f \neq 0$: Stoke (curve integral \rightarrow surface integral)
- C is closed and $\nabla \times f = 0$: 0

- Driving Instruction for Surface Integral

- S is open and $\nabla \cdot f \neq 0$: Param
- S is open and $\nabla \cdot f = 0$: Stoke (surface integral \rightarrow curve integral)
- S is closed and $\nabla \cdot f \neq 0$: Gauss
- S is closed and $\nabla \cdot f = 0$: 0

For each of the following problems:

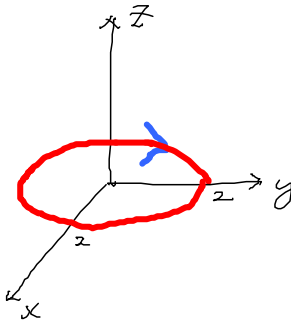
- **a.** Sketch the curve C or the surface S .
State whether it is open or closed.
- b.** Compute $\nabla \times f$ or $\nabla \cdot f$.
State whether it is zero or not.
- c.** State the name of the theorem to be used.
- d.** Apply the theorem (show as much as shown in the class).

3. [1.5 points] $\int_C f \cdot dr$ where

$$f = \langle 2xy^2z^2, 2yz^2 + 2x^2yz^2, 2z + 2y^2z + 2x^2y^2z \rangle$$

$$C = \{(x, y, z) : x^2 + y^2 = 4, z = 0\}, \text{ Oriented counter-clockwise (looking from the below)}$$

a. $C =$



Closed.

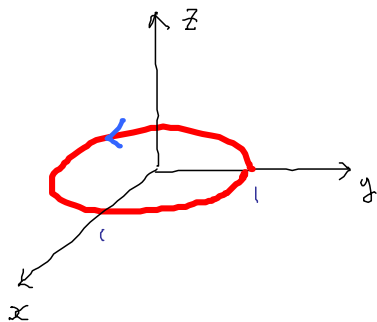
- b.** $\nabla \times f = \langle 0, 0, 0 \rangle = 0$.
- c.** 0
- d.** Nothing to do.

4. [1.5 points] $\int_C f \cdot dr$ where

$$f = \langle xyz, xy, z \rangle$$

$C = \{(x, y, z) : x^2 + y^2 = 1, z = 0\}$, Oriented counter-clockwise (looking from the above)

a. $C =$



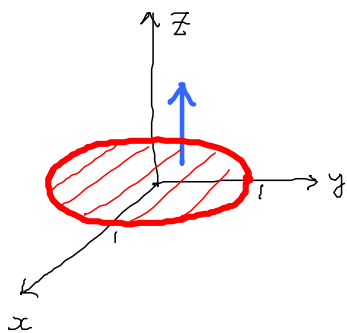
Closed.

b. $\nabla \times f = \langle 0, xy, y - xz \rangle \neq 0.$

c. Stoke's theorem (curve \rightarrow surface)

d. Answer = $\int_S \langle 0, xy, y - xz \rangle \cdot ds$

where $S = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$, Oriented upward.



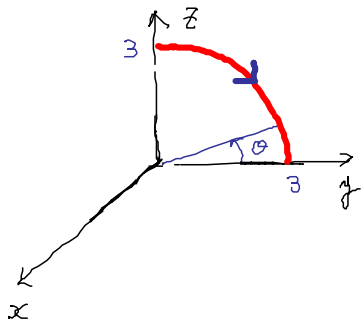
Continue with parametrization.

5. [1.5 points] $\int_C f \cdot dr$ where

$$f = \langle x, xy, xyz \rangle$$

$$C = \{(x, y, z) : y^2 + z^2 = 9, x = 0, y \geq 0, z \geq 0\}, \text{ Oriented clockwise (looking from the front)}$$

a. $C =$



Open.

b. $\nabla \times f = \langle xz, -yz, y \rangle \neq 0.$

c. Parametrization.

d. $r = \langle x, y, z \rangle = \langle 0, 3 \cos \theta, 3 \sin \theta \rangle$

$$\theta_1 = \frac{\pi}{2}$$

$$\theta_2 = 0$$

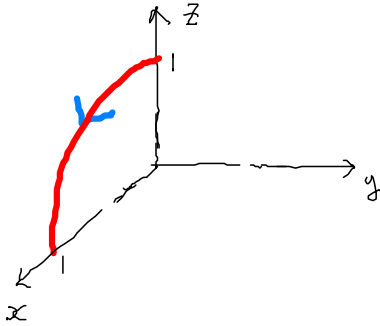
$$\text{Answer} = \int_{\frac{\pi}{2}}^0 \langle 0, 0, 0 \rangle \cdot \langle 0, -3 \sin \theta, 3 \cos \theta \rangle d\theta$$

6. [1.5 points] $\int_C f \cdot dr$ where

$$f = \langle 2xy^2z^2, 2yz^2 + 2x^2yz^2, 2z + 2y^2z + 2x^2y^2z \rangle$$

$C = \{(x, y, z) : x^2 + z^2 = 1, y = 0, x \geq 0, z \geq 0\}$, Oriented counter-clockwise (looking from the right)

a. $C =$



Open.

b. $\nabla \times f = \langle 0, 0, 0 \rangle = 0$

c. Fundamental Theorem of Curve Integral.

d. $r_1 = \langle 0, 0, 1 \rangle$

$r_2 = \langle 1, 0, 0 \rangle$

$$\frac{\partial g}{\partial x} = 2xy^2z^2 \quad (1)$$

$$\frac{\partial g}{\partial y} = 2yz^2 + 2x^2yz^2 \quad (2)$$

$$\frac{\partial g}{\partial z} = 2z + 2y^2z + 2x^2y^2z \quad (3)$$

Integrate (1) in x ,

$$g = x^2y^2z^2 + C(y, z) \quad (4)$$

Plug (4) into (2),

$$2x^2yz^2 + \frac{\partial C}{\partial y} = 2yz^2 + 2x^2yz^2$$

$$\frac{\partial C}{\partial y} = 2yz^2 \quad (5)$$

Integrate (5) in y ,

$$C(y) = y^2z^2 + D(z) \quad (6)$$

Plug (6) into (4),

$$g = x^2y^2z^2 + y^2z^2 + D(z) \quad (7)$$

Plug (7) into (3),

$$2x^2y^2z + 2y^2z + \frac{\partial D}{\partial z} = 2z + 2y^2z + 2x^2y^2z$$

$$\frac{\partial D}{\partial z} = 2z \quad (8)$$

Integrate (8) in z ,

$$D = z^2 + E \quad (9)$$

Plug (9) into (7),

$$g = x^2y^2z^2 + y^2z^2 + z^2 + E$$

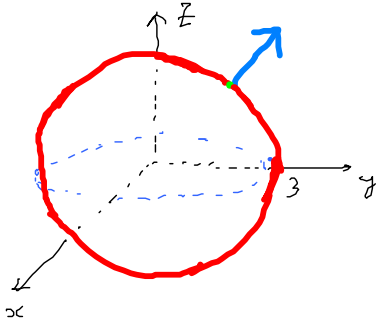
$$\text{Answer} = g(1, 0, 0) - g(0, 0, 1) = 0 - 1 = -1$$

7. [1.5 points] $\int_S f \cdot ds$ where

$$f = \langle xz, -yz, y \rangle$$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 9\}$, The normal direction is away from the origin.

a. $S =$



Closed.

b. $\nabla \cdot f = 0$

c. 0

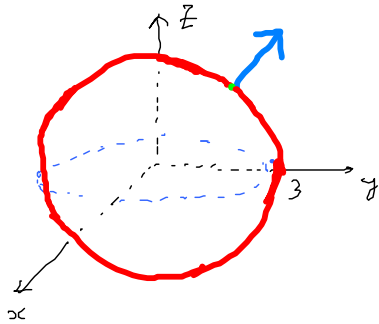
d. Nothing to do.

8. [1.5 points] $\int_S f \cdot ds$ where

$$f = \langle xyz, xy, z \rangle$$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 9\}$, The normal direction is away from the origin.

a. $S =$



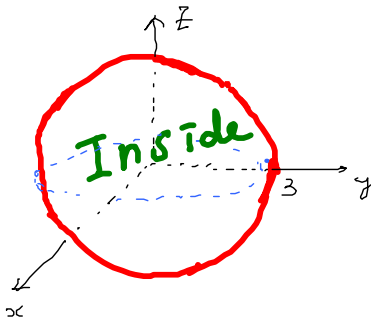
Closed.

b. $\nabla \cdot f = yz + x + 1 \neq 0$

c. Gauss' theorem

d. Answer = $\int_D (yz + x + 1) dV$

where $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\}$

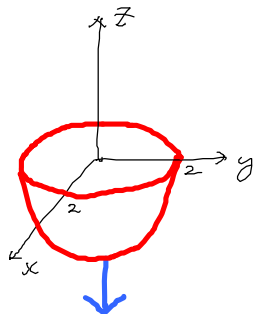


9. [1.5 points] $\int_S f \cdot ds$ where

$$f = \langle xz, -yz, y \rangle$$

$S = \{ (x, y, z) : x^2 + y^2 + z^2 = 4, \quad z \leq 0 \}$, The normal direction is away from the origin.

a. $S =$



Open.

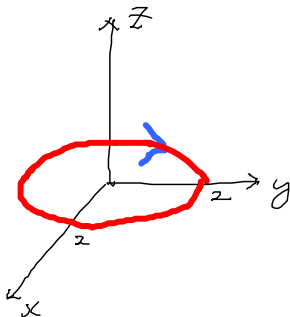
b. $\nabla \cdot f = 0$

c. Stoke's theorem (surface \rightarrow curve)

d. Note that $\nabla \times \langle x, xy, xyz \rangle = \langle xz, -yz, y \rangle$ (From Problem 5).

$$\text{Answer} = \int_C \langle x, xy, xyz \rangle \cdot dr$$

where $C = \{ (x, y, z) : x^2 + y^2 = 4, \quad z = 0 \}$, Oriented clockwise from the above.



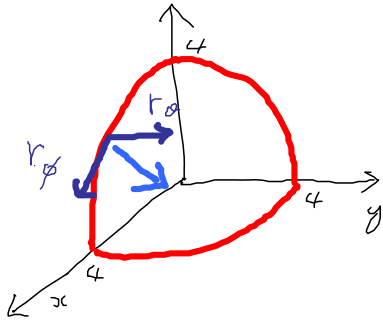
Continue with parametrization.

10. [1.5 points] $\int_S f \cdot ds$ where

$$f = \langle x, z, y \rangle$$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 16, x \geq 0, y \geq 0, z \geq 0\}$, The normal direction is toward the origin.

a. $S =$



Open.

b. $\nabla \cdot f = 1 + 0 + 0 \neq 0$.

c. Surface Parametrization

d. $r = \langle x, y, z \rangle = \langle 4 \sin \phi \cos \theta, 4 \sin \phi \sin \theta, 4 \cos \phi \rangle$

$$\text{Answer} = \int_D \langle 4 \sin \phi \cos \theta, 4 \cos \phi, 4 \sin \phi \sin \theta \rangle \cdot$$

$$\left(\langle -4 \sin \phi \sin \theta, 4 \sin \phi \cos \theta, 0 \rangle \times \langle 4 \cos \phi \cos \theta, 4 \cos \phi \sin \theta, -4 \sin \phi \rangle \right) dA$$

$$\text{where } D = \{(\theta, \phi) : 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}\}$$

