

# MA 242 Test 4

## Multivariate Vector Functions ( $\mathbb{R}^n \rightarrow \mathbb{R}^n$ )

Last Name : \_\_\_\_\_ First Name: \_\_\_\_\_ Student ID: \_\_\_\_\_ Seat Code: \_\_\_\_\_

1. [1.0 points] Write the precise statement of the following theorems:

- Curve Parameterization

$$\int_C f \cdot dr = \int_{t_1}^{t_2} f \cdot \frac{dr}{dt} dt$$

- Surface Parameterization

$$\int_S f \cdot ds = \int_D f \cdot (r_u \times r_v) dA, \text{ RHR}$$

- Fundamental theorem of curve integral

$$\int_C f \cdot dr = [g]_{r_1}^{r_2} \text{ where } \nabla g = f \text{ and } \{r_1, r_2\} = \partial C$$

- Stoke's theorem (curve integral  $\rightarrow$  surface integral)

$$\int_C f \cdot dr = \int_S \nabla \times f \cdot ds \text{ where } C = \partial S, \text{ RHR}$$

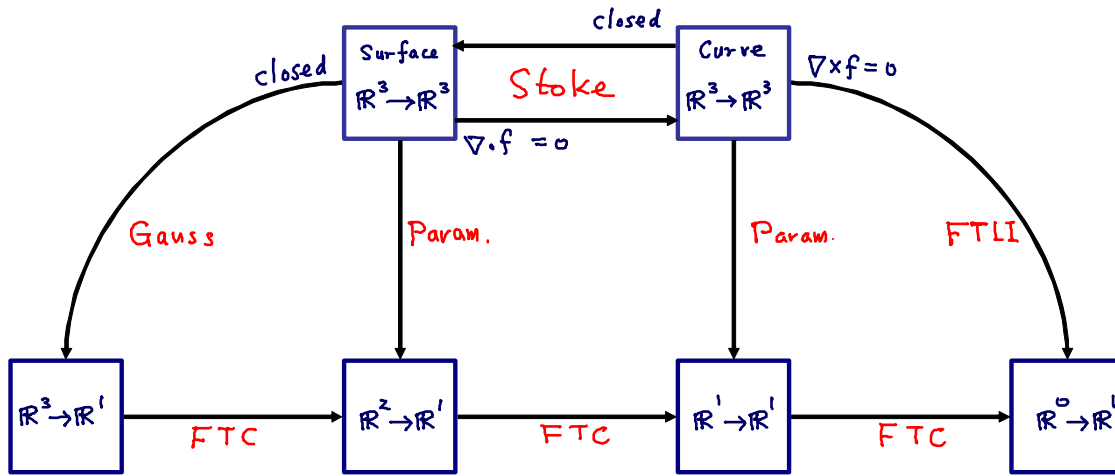
- Stoke's theorem (surface integral  $\rightarrow$  curve integral)

$$\int_S f \cdot ds = \int_C g \cdot dr \text{ where } \nabla \times g = f \text{ and } C = \partial S, \text{ RHR}$$

- Gauss' theorem

$$\int_S f \cdot ds = \int_D \nabla \cdot f dV \text{ where } S = \partial D, \text{ outward}$$

2. [1.0 point] Draw the diagram showing the relationship between various integrations.



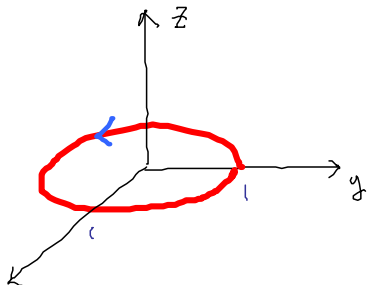
For each of the following six problems do the followings:

- Sketch the curve  $C$  or the surface  $S$
- State whether it is open or closed. Compute  $\nabla \times f$  or  $\nabla \cdot f$ . State whether it is zero or not.
- State the name of the theorem to be used.
- Apply the theorem.

3. [2.0 points]  $\int_C f \cdot dr$  where

$$f = \langle xyz, xy, z \rangle$$

$$C = \{(x, y, z) : x^2 + y^2 = 1, z = 0\}, \text{ Oriented counter-clockwise (looking from the above)}$$



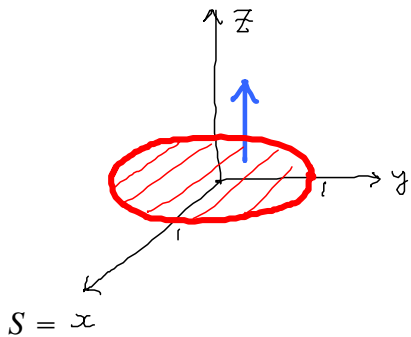
a.  $C = \mathcal{C}$

b. Closed.

$$\nabla \times f = \langle 0, xy, y - xz \rangle \neq 0.$$

c. Use Stoke's theorem

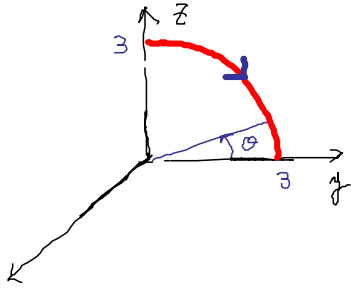
d. Answer =  $\int_S \langle 0, xy, y - xz \rangle \cdot ds$



4. [2.0 points]  $\int_C f \cdot dr$  where

$$f = \langle x, xy, xyz \rangle$$

$C = \{(x, y, z) : y^2 + z^2 = 9, x = 0, y \geq 0, z \geq 0\}$ , Oriented clockwise (looking from the front)



a.  $C = \mathcal{C}$

b. Open

$$\nabla \times f = \langle xz, -yz, y \rangle \neq 0.$$

c. Use Parametrization.

d.  $r = \langle x, y, z \rangle = \langle 0, 3 \cos \theta, 3 \sin \theta \rangle$

$$\theta_1 = \frac{\pi}{2}$$

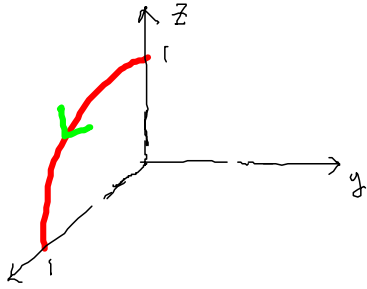
$$\theta_2 = 0$$

$$\text{Answer} = \int_{\frac{\pi}{2}}^0 \langle 0, 0, 0 \rangle \cdot \langle 0, -3 \sin \theta, 3 \cos \theta \rangle d\theta$$

5. [2.0 points]  $\int_C f \cdot dr$  where

$$f = \langle 2xy^2z^2, 2yz^2 + 2x^2yz^2, 2z + 2y^2z + 2x^2y^2z \rangle$$

$C = \{(x, y, z) : x^2 + z^2 = 1, y = 0, x \geq 0, z \geq 0\}$ , Oriented counter-clockwise (looking from the right)



a.  $C = x$

b. Open

$$\nabla \times f = \langle 0, 0, 0 \rangle = 0$$

c. Use Fundamental Theorem of Curve Integral.

d.  $r_1 = \langle 0, 0, 1 \rangle$

$$r_2 = \langle 1, 0, 0 \rangle$$

$$\frac{\partial g}{\partial x} = 2xy^2z^2 \quad (1)$$

$$\frac{\partial g}{\partial y} = 2yz^2 + 2x^2yz^2 \quad (2)$$

$$\frac{\partial g}{\partial z} = 2z + 2y^2z + 2x^2y^2z \quad (3)$$

Integrate (1) in  $x$ ,

$$g = x^2y^2z^2 + C(y, z) \quad (4)$$

Plug (4) into (2),

$$2x^2yz^2 + \frac{\partial C}{\partial y} = 2yz^2 + 2x^2yz^2$$

$$\frac{\partial C}{\partial y} = 2yz^2 \quad (5)$$

Integrate (5) in  $y$ ,

$$C(y) = y^2z^2 + D(z) \quad (6)$$

Plug (6) into (4),

$$g = x^2y^2z^2 + y^2z^2 + D(z) \quad (7)$$

Plug (7) into (3),

$$2x^2y^2z + 2y^2z + \frac{\partial D}{\partial z} = 2z + 2y^2z + 2x^2y^2z$$

$$\frac{\partial D}{\partial z} = 2z \quad (8)$$

Integrate (8) in  $z$ ,

$$D = z^2 + E \quad (9)$$

Plug (9) into (7),

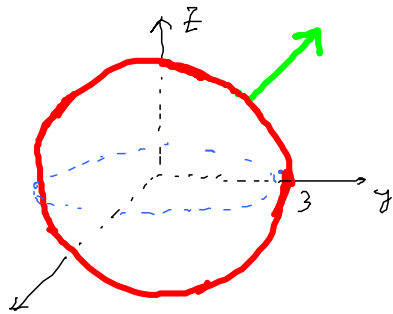
$$g = x^2y^2z^2 + y^2z^2 + z^2 + E$$

$$\text{Answer} = g(1, 0, 0) - g(0, 0, 1) = 0 - 1 = -1$$

6. [2.0 points]  $\int_S f \cdot ds$  where

$$f = \langle xyz, xy, z \rangle$$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 9\}$ , The normal direction is away from the origin.



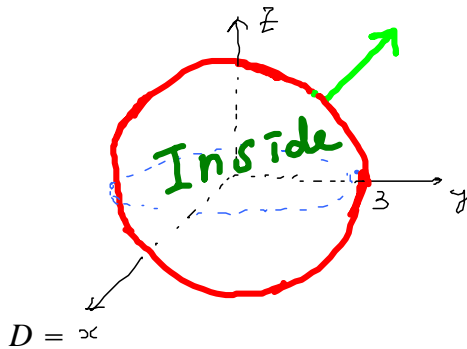
a.  $S = \infty$

b. Closed.

$$\nabla \cdot f = yz + x + 1 \neq 0$$

c. Use Gauss' theorem

d. Answer =  $\int_D (yz + x + 1) dV$

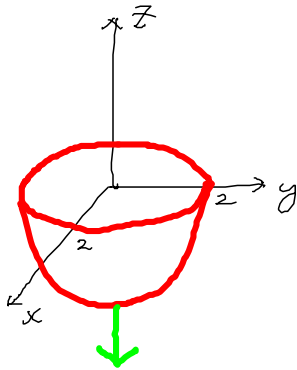


$D = \infty$

7. [2.0 points]  $\int_S f \cdot ds$  where

$$f = \langle xz, -yz, y \rangle$$

$S = \{ (x, y, z) : x^2 + y^2 + z^2 = 4, \quad z \leq 0 \}$ , The normal direction is away from the origin.



a.  $S =$

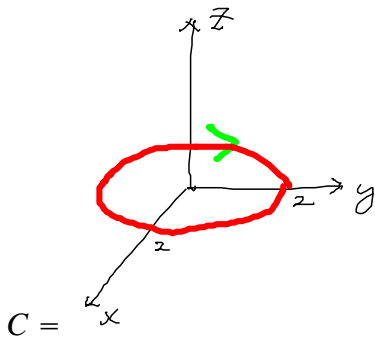
b. Open

$$\nabla \cdot f = 0$$

c. Use Stoke's theorem

d. Note that  $\nabla \times \langle x, xy, xyz \rangle = \langle xz, -yz, y \rangle$  (From Problem 4).

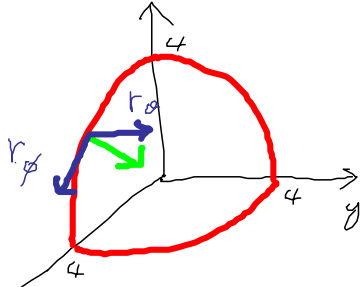
$$\text{Answer} = \int_C \langle x, xy, xyz \rangle \cdot dr$$



8. [2.0 points]  $\int_S f \cdot ds$  where

$$f = \langle x, xy, xyz \rangle$$

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 16, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0\}$ , The normal direction is toward the origin.



a.  $S = \leftarrow x$

b. Open.

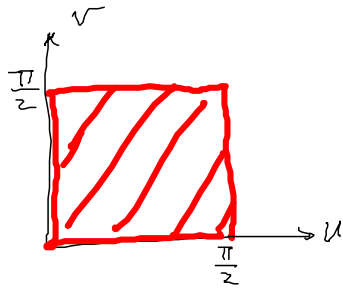
$$\nabla \cdot f = 1 + x + xy \neq 0.$$

c. Use Parametrization

d.  $r = \langle x, y, z \rangle = \langle 4 \sin v \cos u, 4 \sin v \sin u, 4 \cos v \rangle$

$$\text{Answer} = \int_D \langle 4 \sin v \cos u, 4^2 \sin^2 v \cos u \sin u, 4^3 \sin^2 v \cos u \sin u \cos v \rangle \cdot$$

$$\left( \langle -4 \sin v \sin u, 4 \sin v \cos u, 0 \rangle \times \langle 4 \cos v \cos u, 4 \cos v \sin u, -4 \sin v \rangle \right) dA$$



$D =$