# MA 242 Test 2

## MISO ($\mathbb{R}^n \to \mathbb{R}^1$): Differentiation

**1.** [1.5 point] Write the definition for each of the followings:
($f(x_1, x_2, x_3)$ is a MISO.)

- $\frac{\partial f}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2, x_3) - f(x_1, x_2, x_3)}{\Delta x_2}$
- $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle$
- $D_u f = \lim_{\Delta t \to 0} \frac{f\left(x + \Delta t\, u\right) - f(x)}{\Delta t}$

**2.** [1.5 point] Let $z = x \ln(x + 2y)$ where $x = \sin t$ and $y = \cos t$. Find $\frac{dz}{dt}$.

**Theorem:** $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\frac{dz}{dt} = \left(\ln(x + 2y) + x\frac{1}{x+2y}\right)\cos t + \left(x\frac{1}{x+2y}2\right)(-\sin t).$

**3.** [1.5 point] Let $z = x^2 y$ where $x = st$ and $y = s^2 + t^2$. Find $\frac{\partial z}{\partial t}$.

**Theorem:** $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$\frac{\partial z}{\partial t} = (2xy)(s) + (x^2)(2t).$

**4.** [1.5 point] Let $y(x)$ the function define by $\sin x + \cos y - \sin x \cos y = 0$. Find $\frac{dy}{dx}$.

**Theorem:** $\frac{dy}{dx} = -\frac{F_x}{F_y}$

$\frac{dy}{dx} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y}$

**5.** [1.5 point] Let $f(x,y) = x^3 y^4$. Find the maximum slope of $f$ at $(1, -1)$. Find the direction.

**Theorem:** the maximum slope $= |\nabla f|$ and the direction $= \frac{\nabla f}{|\nabla f|}$.

$\nabla f = \langle 3x^2 y^4, 4x^3 y^3 \rangle$

$\nabla f(1, -1) = \langle 3, -4 \rangle$

The maximum slope $= \sqrt{3^2 + (-4)^2} = 5$

The direction $= \frac{\langle 3, -4 \rangle}{5} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

**6.** [2.0 point] Let $f(x,y) = xy^2 + x^2 y^3$. Find the directional derivative of $f$ at $(1, -1)$ in the direction of the vector $\langle 5, 12 \rangle$.

**Theorem:** $D_u f = \nabla f \bullet u$

$\nabla f = \langle y^2 + 2xy^3, 2xy + 3x^2 y^2 \rangle$

$\nabla f(1, -1) = \langle -1, 1 \rangle$

$u = \frac{\langle 5, 12 \rangle}{\sqrt{5^2 + 12^2}} = \frac{\langle 5, 12 \rangle}{13}$

$D_u f(1, -1) = \langle -1, 1 \rangle \bullet \frac{\langle 5, 12 \rangle}{13} = \frac{7}{13}$

**7.** [3 point] Let $f(x,y) = e^x y^2$. Find an approximate value of $f(-0.1, 1.1)$ via the linear approximation method.

**Theorem:** $\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$

$\Delta f \approx (e^x y^2)\Delta x + (2e^x y)\Delta y$

We choose $x = 0, y = 1$.

$\Delta x = -0.1,\ \Delta y = 0.1$

$\Delta f \approx (e^0 1^2)(-0.1) + (2e^0 1)(0.1) = 0.1$

$f(-0.1, 1.1) \approx f(0,1) + \Delta f = e^0 1^2 + 0.1 = 1.1$

**8.** [3 points] Let $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$. It is known that it has the following critical points:

$$(-1,2), (-1,-2), (0,0), \left(-\frac{5}{3}, 0\right).$$

Classify them into local maximum, local minimum or saddle.

**Theorem:** Let $D = f_{xx}f_{yy} - f_{xy}^2$. Then

(a) If $D > 0$ and $f_{xx} > 0$ then local min
(b) If $D > 0$ and $f_{xx} < 0$ then local max
(c) If $D < 0$ then saddle
(d) If $D = 0$ then don't know.

$f_{xx} = 12x + 10$

$f_{yy} = 2x + 2$

$f_{xy} = 2y$

$D = (12x + 10)(2x + 2) - (2y)^2$

| $p$ | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | $D$ | type |
|---|---|---|---|---|---|
| $(-1,2)$ | $-2$ | $0$ | $4$ | $-16$ | saddle |
| $(-1,-2)$ | $-2$ | $0$ | $-4$ | $-16$ | saddle |
| $(0,0)$ | $10$ | $2$ | $0$ | $20$ | local min |
| $\left(-\frac{5}{3},0\right)$ | $-10$ | $-\frac{4}{3}$ | $0$ | $\frac{40}{3}$ | local max |

**9.** [6 points] Find the maximum and the minimum of the function $f(x, y) = 2x^2 + 3y^2 - 4x + 5$ subject to $g(x, y) = x^2 + 2y^2 - 4 \leq 0$.

**Subproblem:** Find the critical points of $f$ under $g < 0$.

**Theorem:** $\nabla f = 0$ and $g < 0$

$$\begin{cases} 4x - 4 = 0 \\ 6y = 0 \\ x^2 + 2y^2 - 4 < 0 \end{cases}$$

$$\begin{cases} x = 1 \\ y = 0 \\ x^2 + 2y^2 - 4 < 0 \end{cases}$$

$$\begin{cases} x = 1 \\ y = 0 \\ (1)^2 + 2(0)^2 - 4 < 0 \end{cases}$$

$(1, 0)$

**Subproblem:** Find the critical points of $f$ under $g = 0$.

**Theorem:** $\nabla f = \lambda \nabla g$ and $g = 0$

$$\begin{cases} 4x - 4 = \lambda 2x \\ 6y = \lambda 4y \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ 6y - \lambda 4y = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y(6 - 4\lambda) = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases} \quad \text{or} \quad \begin{cases} 4x - 4 - \lambda 2x = 0 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x^2 - 4 = 0 \end{cases} \quad \text{or} \quad \begin{cases} 4x - 4 - 3x = 0 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ (x - 2)(x + 2) = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 4 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x = 2 \end{cases} \quad \text{or} \quad \begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x = -2 \end{cases} \quad \text{or} \quad \begin{cases} x = 4 \\ \lambda = \frac{3}{2} \\ 2y^2 + 12 = 0 \end{cases}$$

$$\begin{cases} \lambda = 1 \\ y = 0 \\ x = 2 \end{cases} \quad \text{or} \quad \begin{cases} \lambda = 3 \\ y = 0 \\ x = -2 \end{cases} \quad \text{None}$$

$(2, 0)$ $(-2, 0)$

**Combine:**

| $p$ | $f$ | type |
|---|---|---|
| $(1,0)$ | 3 | min |
| $(2,0)$ | 5 | |
| $(-2,0)$ | 21 | max |