

# MA 242 Test 2

## Multivariate Scalar Functions ( $\mathbb{R}^n \rightarrow \mathbb{R}^1$ ): Differentiation

1. [1 point] Let  $z = x \ln(x + 2y)$  where  $x = \sin t$  and  $y = \cos t$ . Find  $\frac{dz}{dt}$ .

**Theorem:**  $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

$$\frac{dz}{dt} = \left( \ln(x + 2y) + x \frac{1}{x+2y} \right) \cos t + \left( x \frac{1}{x+2y} 2 \right) (-\sin t).$$

2. [1 point] Let  $z = x^2y$  where  $x = st$  and  $y = s^2 + t^2$ . Find  $\frac{\partial z}{\partial t}$ .

**Theorem:**  $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

$$\frac{\partial z}{\partial t} = (2xy)(s) + (x^2)(2t).$$

3. [1 point] Let  $y(x)$  the function define by  $\sin x + \cos y - \sin x \cos y = 0$ . Find  $\frac{dy}{dx}$ .

**Theorem:**  $\frac{dy}{dx} = -\frac{F_x}{F_y}$

$$\frac{dy}{dx} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y}$$

4. [1.5 point] Let  $f(x, y) = x^3y^4$ . Find the maximum slope of  $f$  at  $(1, -1)$ . Find the direction.

**Theorem:** the maximum slope =  $|\vec{\nabla}f|$  and the direction =  $\frac{\vec{\nabla}f}{|\vec{\nabla}f|}$ .

$$\vec{\nabla}f = \langle 3x^2y^4, 4x^3y^3 \rangle$$

$$\vec{\nabla}f(1, -1) = \langle 3, -4 \rangle$$

$$\text{The maximum slope} = \sqrt{3^2 + (-4)^2} = 5$$

$$\text{The direction} = \frac{\langle 3, -4 \rangle}{5} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$$

5. [1.5 point] Let  $f(x, y) = xy^2 + x^2y^3$ . Find the directional derivative of  $f$  at  $(1, -1)$  in the direction of the vector  $\langle 5, 12 \rangle$ .

**Theorem:**  $D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u}$

$$\vec{\nabla}f = \langle y^2 + 2xy^3, 2xy + 3x^2y^2 \rangle$$

$$\vec{\nabla}f(1, -1) = \langle -1, 1 \rangle$$

$$\vec{u} = \frac{\langle 5, 12 \rangle}{\sqrt{5^2 + 12^2}} = \frac{\langle 5, 12 \rangle}{13}$$

$$D_{\vec{u}}f(1, -1) = \langle -1, 1 \rangle \cdot \frac{\langle 5, 12 \rangle}{13} = \frac{7}{13}$$

6. [2 point] Let  $f(x,y) = e^xy^2$ . Find an approximate value of  $f(-0.1, 1.1)$  via the linear approximation method.

**Theorem:**  $\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$

$$\Delta f \approx (e^xy^2)\Delta x + (2e^xy)\Delta y$$

We choose  $x = 0, y = 1$ .

$$\Delta x = -0.1, \Delta y = 0.1$$

$$\Delta f \approx (e^0 1^2)(-0.1) + (2e^0 1)(0.1) = 0.1$$

$$f(-0.1, 1.1) \approx f(0, 1) + \Delta f = e^0 1^2 + 0.1 = 1.1$$

7. [2 points] Let  $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ . It is known that it has the following critical points:

$$(-1, 2), (-1, -2), (0, 0), \left(-\frac{5}{3}, 0\right).$$

Classify them into local maximum, local minimum or saddle.

**Theorem:** Let  $D = f_{xx}f_{yy} - f_{xy}^2$ . Then

- (a) If  $D > 0$  and  $f_{xx} > 0$  then local min
- (b) If  $D > 0$  and  $f_{xx} < 0$  then local max
- (c) If  $D < 0$  then saddle
- (d) Otherwise then don't know.

$$f_{xx} = 12x + 10$$

$$f_{yy} = 2x + 2$$

$$f_{xy} = 2y$$

$$D = (12x + 10)(2x + 2) - (2y)^2$$

$x$	$y$	$f_{xx}$	$f_{yy}$	$f_{xy}$	$D$	type
-1	2	-2	0	4	-16	saddle
-1	-2	-2	0	-4	-16	saddle
0	0	10	2	0	20	local min
$-\frac{5}{3}$	0	-10	$-\frac{4}{3}$	0	$\frac{40}{3}$	local max

8. [4 points] Find the maximum and the minimum of the function  $f(x,y) = 2x^2 + 3y^2 - 4x + 5$  subject to  $g(x,y) = x^2 + 2y^2 - 4 \leq 0$ .

**Subproblem:** Find the critical points of  $f$  under  $g = 0$ .

**Theorem:**  $\vec{\nabla}f = \lambda \vec{\nabla}g$  and  $g = 0$

$$\begin{cases} 4x - 4 = \lambda 2x \\ 6y = \lambda 4y \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ 6y - \lambda 4y = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y(6 - 4\lambda) = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x^2 - 4 = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ (x - 2)(x + 2) = 0 \end{cases}$$

$$\begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x = 2 \end{cases}$$

$$\begin{cases} \lambda = 1 \\ y = 0 \\ x = 2 \end{cases}$$

$(2,0)$

$$\text{or } \begin{cases} 4x - 4 - \lambda 2x = 0 \\ y = 0 \\ x = -2 \end{cases}$$

$$\text{or } \begin{cases} \lambda = 3 \\ y = 0 \\ x = -2 \end{cases}$$

$(-2,0)$

$$\text{or } \begin{cases} 4x - 4 - \lambda 2x = 0 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\text{or } \begin{cases} 4x - 4 - 3x = 0 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\text{or } \begin{cases} x = 4 \\ \lambda = \frac{3}{2} \\ x^2 + 2y^2 - 4 = 0 \end{cases}$$

$$\text{or } \begin{cases} x = 4 \\ \lambda = \frac{3}{2} \\ 2y^2 + 12 = 0 \end{cases}$$

None

**Subproblem:** Find the critical points of  $f$  under  $g < 0$ .

**Theorem:**  $\vec{\nabla}f = 0$  and  $g < 0$

$$\begin{cases} 4x - 4 = 0 \\ 6y = 0 \\ x^2 + 2y^2 - 4 < 0 \end{cases}$$

$$\begin{cases} x = 1 \\ y = 0 \\ x^2 + 2y^2 - 4 < 0 \end{cases}$$

$$\begin{cases} x = 1 \\ y = 0 \\ (1)^2 + 2(0)^2 - 4 < 0 \end{cases}$$

$(1,0)$

$x$	$y$	$f$	type
2	0	5	
-2	0	21	max
1	0	3	min