

# MA 242 Test 1

## Univariate Vector Functions ( $\mathbb{R}^1 \rightarrow \mathbb{R}^n$ )

Last Name : \_\_\_\_\_ First Name: \_\_\_\_\_ Student ID: \_\_\_\_\_ Seat Code: \_\_\_\_\_

1. (1 points) Determine  $x$  such that the two vectors  $\vec{a} = \langle -5, 3, x \rangle$  and  $\vec{b} = \langle 6, -8, 2 \rangle$  are orthogonal.

● Use the formula:  $\vec{a} \perp \vec{b}$  iff  $\vec{a} \cdot \vec{b} = 0$ :

$$\begin{aligned}0 &= \vec{a} \cdot \vec{b} \\ &= \langle -5, 3, x \rangle \cdot \langle 6, -8, 2 \rangle \\ &= (-5)(6) + (3)(-8) + (x)(2) \\ &= -30 - 24 + 2x \\ &= -54 + 2x \\ x &= 27.\end{aligned}$$

2. (1 points) Determine  $x$  and  $y$  such that the two vectors  $\vec{a} = \langle 2, 3, x+y \rangle$  and  $\vec{b} = \langle 6, x, -12 \rangle$  are parallel.

● Use the formula:  $\vec{a} \parallel \vec{b}$  iff  $\exists c \vec{b} = c\vec{a}$ : Note

$$\begin{aligned}6 &= c2 \\ x &= c3 \\ -12 &= c(x+y)\end{aligned}$$

Hence

$$c = 3$$

Thus

$$\begin{aligned}x &= 9 \\ y &= -13\end{aligned}$$

3. (1 points) Find a unit vector  $\vec{c}$  which is orthogonal to both  $\vec{a} = \langle -1, 1, -1 \rangle$  and  $\vec{b} = \langle 2, 1, 3 \rangle$ .

● Use the formula:  $\vec{c} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$\begin{aligned}\vec{a} \times \vec{b} &= \langle -1, 1, -1 \rangle \times \langle 2, 1, 3 \rangle \\ &= \langle (1)(3) - (-1)(1), (-1)(2) - (-1)(3), (-1)(1) - (1)(2) \rangle \\ &= \langle 4, 1, -3 \rangle \\ \vec{c} &= \frac{\langle 4, 1, -3 \rangle}{\sqrt{4^2 + 1^2 + (-3)^2}} \\ &= \frac{\langle 4, 1, -3 \rangle}{\sqrt{26}}\end{aligned}$$

4. (1 points) Find the area  $A$  of the parallelogram determined by  $\vec{a} = \langle -3, -2, 4 \rangle$  and  $\vec{b} = \langle -1, 2, 1 \rangle$

● Use the formula:  $A = |\vec{a} \times \vec{b}|$

$$\begin{aligned}
\vec{a} \times \vec{b} &= \langle -3, -2, 4 \rangle \times \langle -1, 2, 1 \rangle \\
&= \langle (-2)(1) - (4)(2), (4)(-1) - (-3)(1), (-3)(2) - (-2)(-1) \rangle \\
&= \langle -10, -1, -8 \rangle \\
|\vec{a} \times \vec{b}| &= \sqrt{(-10)^2 + (-1)^2 + (-8)^2} \\
&= \sqrt{165}
\end{aligned}$$

5. (1 points) Find the volume  $V$  of the parallelepiped determined by  $\vec{a} = \langle 6, 3, -1 \rangle$ ,  $\vec{b} = \langle 0, 1, 2 \rangle$  and  $\vec{c} = \langle -4, 2, -5 \rangle$

● Use the formula:  $V = |\vec{a} \cdot \vec{b} \times \vec{c}|$

$$\begin{aligned}
|\vec{a} \cdot \vec{b} \times \vec{c}| &= (6)((1)(-5) - (2)(2)) + (3)((2)(-4) - (0)(-5)) + (-1)((0)(2) - (1)(-4)) \\
&= -82
\end{aligned}$$

$$V = |-82| = 82$$

6. (2 points) Find the length  $L$  of the arc given by  $\vec{r}(t) = \langle \frac{t^2}{2}, \sqrt{2}t, \ln t \rangle$  where  $1 \leq t \leq e$ .

● Use the formula:  $L = \int_a^b |\vec{r}'(t)| dt$

$$\vec{r}' = \langle t, \sqrt{2}, t^{-1} \rangle$$

$$|\vec{r}'| = \sqrt{t^2 + 2 + t^{-2}} = \sqrt{(t + t^{-1})^2} = t + t^{-1}$$

$$L = \int_1^e t + t^{-1} dt = \left[ \frac{t^2}{2} + \ln t \right]_1^e = \left( \frac{e^2}{2} + \ln e \right) - \left( \frac{1^2}{2} + \ln 1 \right) = \left( \frac{e^2}{2} + 1 \right) - \left( \frac{1}{2} + 0 \right) = \frac{e^2}{2} + \frac{1}{2}$$

7. (2 points) Find the position  $\vec{r}(t)$  and the velocity  $\vec{v}(t)$  of a rocket whose acceleration is given by

$$\vec{a} = \langle 4t, 6t, 1 \rangle$$

and whose initial conditions are given by

$$\vec{r}(0) = \langle 1, 0, 0 \rangle$$

$$\vec{v}(0) = \langle 1, -1, 1 \rangle$$

● Use the formula:  $\vec{v}(t) = \int \vec{a}(t) dt$  and  $\vec{r}(t) = \int \vec{v}(t) dt$ .

$$\vec{v}(t) = \int \langle 4t, 6t, 1 \rangle dt = \langle 2t^2, 3t^2, t \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{C} = \langle 1, -1, 1 \rangle$$

$$\vec{C} = \langle 1, -1, 1 \rangle$$

$$\vec{v}(t) = \langle 2t^2, 3t^2, t \rangle + \langle 1, -1, 1 \rangle = \langle 2t^2 + 1, 3t^2 - 1, t + 1 \rangle$$

$$\vec{r}(t) = \int \langle 2t^2 + 1, 3t^2 - 1, t + 1 \rangle dt = \left\langle \frac{2}{3}t^3 + t, t^3 - t, \frac{t^2}{2} + t \right\rangle + \vec{C}$$

$$\vec{r}(0) = \langle 0, 0, 0 \rangle + \vec{C} = \langle 1, 0, 0 \rangle$$

$$\vec{C} = \langle 1, 0, 0 \rangle$$

$$\vec{r}(t) = \left\langle \frac{2}{3}t^3 + t, t^3 - t, \frac{t^2}{2} + t \right\rangle + \langle 1, 0, 0 \rangle = \left\langle \frac{2}{3}t^3 + t + 1, t^3 - t, \frac{t^2}{2} + t \right\rangle$$

8. (3 points) Find the tangent  $\vec{T}$ , normal  $\vec{N}$  and binormal  $\vec{B}$  vectors for

$$\vec{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$$

for  $t > 0$ .

- Use the formula:  $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ ,  $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$  and  $\vec{B} = \vec{T} \times \vec{N}$ .

$$\vec{T} = \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{(2t)^2 + (t \sin t)^2 + (t \cos t)^2}} = \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{5t^2}} = \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{5}t} = \frac{\langle 2, \sin t, \cos t \rangle}{\sqrt{5}}$$

$$\vec{N} = \frac{\frac{\langle 0, \cos t, -\sin t \rangle}{\sqrt{5}}}{\frac{\sqrt{0^2 + \cos^2 t + \sin^2 t}}{\sqrt{5}}} = \frac{\langle 0, \cos t, -\sin t \rangle}{\sqrt{0^2 + \cos^2 t + \sin^2 t}} = \langle 0, \cos t, -\sin t \rangle$$

$$\begin{aligned} \vec{B} &= \frac{\langle 2, \sin t, \cos t \rangle}{\sqrt{5}} \times \langle 0, \cos t, -\sin t \rangle \\ &= \frac{\langle \sin t(-\sin t) - \cos t \cos t, \cos t(0) - 2(-\sin t), 2(\cos t) - \sin t(0) \rangle}{\sqrt{5}} \\ &= \frac{\langle -1, 2 \sin t, 2 \cos t \rangle}{\sqrt{5}} \end{aligned}$$

9. (1 point) Let  $\vec{r}(t) = \langle t, \cos t, \sin t \rangle$ . It is known that it has the moving frame

$$\vec{T} = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle, \quad \vec{N} = \langle 0, -1, 0 \rangle, \quad \vec{B} = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$$

at time 0. Find the tangential  $a_T$ , normal  $a_N$ , and binormal  $a_B$  components of the acceleration at time 0.

- Use the formula:  $a_T = \vec{a} \cdot \vec{T}$ ,  $a_N = \vec{a} \cdot \vec{N}$ ,  $a_B = \vec{a} \cdot \vec{B}$

$$\vec{a} = \langle 0, -\cos t, -\sin t \rangle$$

$$\vec{a}(0) = \langle 0, -1, 0 \rangle$$

$$a_T = \langle 0, -1, 0 \rangle \cdot \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}} = 0$$

$$a_N = \langle 0, -1, 0 \rangle \cdot \langle 0, -1, 0 \rangle = 1$$

$$a_B = \langle 0, -1, 0 \rangle \cdot \frac{\langle 1, 0, -1 \rangle}{\sqrt{2}} = 0$$

10. (1 point) Let  $\vec{r}(t) = \langle 1, t, t^2 \rangle$ . It is known that  $\vec{T}'(0) = \langle 0, 0, 2 \rangle$ . Find the curvature  $\kappa$  at  $t = 0$ .

- Use the formula:  $\kappa = \frac{|\vec{T}'|}{|\vec{r}'|}$

$$\vec{r}' = \langle 0, 1, 2t \rangle$$

$$\vec{r}'(0) = \langle 0, 1, 0 \rangle$$

$$\kappa(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|} = \frac{|\langle 0, 0, 2 \rangle|}{|\langle 0, 1, 0 \rangle|} = \frac{2}{1} = 2$$