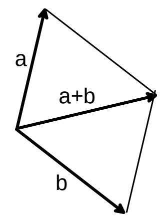
MA 242 Test 1

SIMO: $\mathbb{R}^1 \to \mathbb{R}^n$

Last Name : ____________ First Name: __________ Student ID: _________ Seat Code: _____

1. (2.0 point) Write the definition for each of the followings
(a, b are vectors, c is a positive scalar,$a(t)$ is a SIMO.)

- $|a|$ is the length of the vector a.
- $a + b$ (Draw the definition)

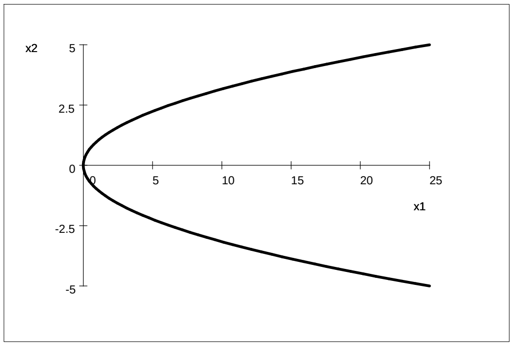


- ca is the vector whose magintude is $c|a|$ and direction is the same as a.
- $a \cdot b = |a||b|\cos\theta$
- $a \times b$ is the vector who mangnitude is $|a||b|\sin\theta$ and direction is given according to *RHR* on a and b.
- $\frac{d}{dt}a(t) = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t}$
- $\int_{t_1}^{t_2} a(t)dt = \lim_{\Delta t_i \to 0} \sum_i a(t_i)\Delta t_i$

2. (2.0 point) Write a theorem for each of the followings:

- $|\langle a_1, a_2, a_3 \rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- $c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$
- $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1b_1 + a_2b_2 + a_3b_3$
- $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$
- $\frac{d}{dt}\langle a_1(t), a_2(t), a_3(t) \rangle = \left\langle \frac{d}{dt}a_1(t), \frac{d}{dt}a_2(t), \frac{d}{dt}a_3(t) \right\rangle$
- $\int \langle a_1(t), a_2(t), a_3(t) \rangle dt = \left\langle \int a_1(t)dt, \int a_2(t)dt, \int a_3(t)dt \right\rangle$

3. (1.0 point) Sketch the function $f(t) = \langle t^2, t \rangle$.



4. (1.5 point) Determine whether the two vectors $a = \langle -5, 3, 1 \rangle$ and $b = \langle 6, -8, 2 \rangle$ are orthogonal.

- Use the formula: $a \perp b$ iff $a \cdot b = 0$:

$$
\begin{aligned}
0 &= a \cdot b \\
&= \langle -5, 3, 1 \rangle \cdot \langle 6, -8, 2 \rangle \\
&= (-5)(6) + (3)(-8) + (1)(2) \\
&= -52 \\
&\neq 0
\end{aligned}
$$

No.

5. (1.5 point) Find the unit vector u that has the same direction as $a = \langle 1, 2, 3 \rangle$.

- Use the formula: $u = \frac{a}{|a|}$

$$
u = \frac{\langle 1, 2, 3 \rangle}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}}
$$

6. (1.5 point) Find a vector c which is orthogonal to both $a = \langle -1, 1, -1 \rangle$ and $b = \langle 2, 1, 3 \rangle$.

- Use the formula: $c = a \times b$

$$
\begin{aligned}
c &= \langle -1, 1, -1 \rangle \times \langle 2, 1, 3 \rangle \\
&= \langle (1)(3) - (-1)(1), (-1)(2) - (-1)(3), (-1)(1) - (1)(2) \rangle \\
&= \langle 4, 1, -3 \rangle
\end{aligned}
$$

7. (1.5 point) Find the area A of the parallelogram determined by $a = \langle -3, -2, 4 \rangle$ and $b = \langle -1, 2, 1 \rangle$

- Use the formula: $A = |a \times b|$

$$
\begin{aligned}
A &= \left| \langle -3, -2, 4 \rangle \times \langle -1, 2, 1 \rangle \right| \\
&= \left| \left\langle (-2)(1) - (4)(2), (4)(-1) - (-3)(1), (-3)(2) - (-2)(-1) \right\rangle \right| \\
&= \left| \langle -10, -1, -8 \rangle \right| \\
&= \sqrt{(-10)^2 + (-1)^2 + (-8)^2} \\
&= \sqrt{165}
\end{aligned}
$$

8. (1.5 point) Find the volume V of the parallelepiped determined by $a = \langle 1, 2, 3 \rangle$, $b = \langle 1, 2, 1 \rangle$ and $c = \langle 3, 2, 1 \rangle$

- Use the formula: $V = |a \cdot (b \times c)|$

$$
\begin{aligned}
V &= |\langle 1, 2, 3 \rangle \cdot (\langle 1, 2, 1 \rangle \times \langle 3, 2, 1 \rangle)| \\
&= |((1)(2)(1) + (2)(1)(3) + (1)(2)(3)) - ((3)(2)(3) + (2)(1)(1) + (1)(2)(1))| \\
&= |-8| = 8
\end{aligned}
$$

9. (1.5 point) Find the position $r(t)$ of a rocket whose velocity is given by $v = \langle t, \cos t, e^t \rangle$ and whose position at time 0 is given by $r(0) = \langle 0, 1, 0 \rangle$.

- Use the formula: $r(t) = \int v(t)\, dt$.

$$r(t) = \int \langle t, \cos t, e^t \rangle dt = \left\langle \frac{t^2}{2}, \sin t, e^t \right\rangle + C$$

$$r(0) = \langle 0,0,1 \rangle + C = \langle 0,1,0 \rangle$$

$$C = \langle 0,1,-1 \rangle$$

$$r(t) = \left\langle \frac{t^2}{2}, \sin t, e^t \right\rangle + \langle 0,1,-1 \rangle = \left\langle \frac{t^2}{2}, \sin t + 1, e^t - 1 \right\rangle$$

10. (1.5 point) Find the length L of the arc given by $r(t) = \left\langle \frac{1}{2}t^2, \sqrt{2}\,t, \ln t \right\rangle$ where $1 \le t \le e$.

- Use the formula: $L = \int_a^b |r'(t)| dt$

$$r' = \left\langle t, \sqrt{2}, t^{-1} \right\rangle$$

$$|r'| = \sqrt{t^2 + 2 + t^{-2}} = \sqrt{(t + t^{-1})^2} = t + t^{-1}$$

$$L = \int_1^e t + t^{-1} dt = \left[\frac{t^2}{2} + \ln t \right]_1^e = \left(\frac{e^2}{2} + \ln e \right) - \left(\frac{1^2}{2} + \ln 1 \right) = \left(\frac{e^2}{2} + 1 \right) - \left(\frac{1}{2} + 0 \right)$$

$$= \frac{e^2}{2} + \frac{1}{2}$$

11. (1.5 point) Consider a space cruve given by $r(t) = \langle t, t^2, t^3 \rangle$. Find the tangent vector T at $t = 0$.

- Use the formula: $T = \frac{r'}{|r'|}$

$$T = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{(1)^2 + (2t)^2 + (3t^2)^2}} = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}}$$

$$T(0) = \frac{\langle 1,0,0 \rangle}{\sqrt{1}} = \langle 1,0,0 \rangle$$

12. (1.5 point) Suppose that a space curve has the tangent vector $T = \frac{1}{\sqrt{2}} \langle \cos t, 1, \sin t \rangle$. Find the normal vector N at $t = 0$.

- Use the formula: $N = \frac{T'}{|T'|}$

$$N = \frac{\frac{1}{\sqrt{2}} \langle -\sin t, 0, \cos t \rangle}{\frac{1}{\sqrt{2}} \sqrt{(-\sin t)^2 + (0)^2 + (\cos t)^2}} = \frac{\frac{1}{\sqrt{2}} \langle -\sin t, 0, \cos t \rangle}{\frac{1}{\sqrt{2}}} = \langle -\sin t, 0, \cos t \rangle$$

$$N(0) = \langle 0,0,1 \rangle$$

13. (1.5 point) Suppose that a space curve has the tangent $T = \frac{1}{\sqrt{21}} \langle 1,2,4 \rangle$ and the normal $N = \frac{1}{\sqrt{6}} \langle 2,1,-1 \rangle$ at $t = 0$. Find the binormal vector B at $t = 0$.

- Use the formula: $B = T \times N$

$$B = \frac{\langle 1,2,4 \rangle}{\sqrt{21}} \times \frac{\langle 2,1,-1 \rangle}{\sqrt{6}} = \frac{\langle -6,9,-3 \rangle}{\sqrt{126}}$$

14. (1.5 point) Let $r(t) = \langle 1, t, t^2 \rangle$. It is known that $T'(0) = \langle 0,0,2 \rangle$. Find the curvature κ at $t = 0$.

- Use the formula: $\kappa = \frac{|T'|}{|r'|}$

$$
\begin{aligned}
r' &= \langle 0, 1, 2t \rangle \\
r'(0) &= \langle 0, 1, 0 \rangle \\
\kappa(0) &= \frac{|T'(0)|}{|r'(0)|} = \frac{|\langle 0, 0, 2 \rangle|}{|\langle 0, 1, 0 \rangle|} = \frac{2}{1} = 2
\end{aligned}
$$