Abstract—In order to improve the spectrum efficiency, spectrum sharing systems allow multiple systems to utilize the same spectrum with different priorities. Typically, the primary network performs as a stand-alone network while the secondary one accesses the spectrum only if it does no harm to the primary receivers. In this paper, the configuration of the secondary network is of our interest and we explore its single-hop transport throughput (STT) with outage constraints imposed on both networks. STT is a new metric that inherits the merits of both the traditional transport capacity and transmission capacity, incorporating transmission distance and outage probability into a uniform framework. Given the settings of the primary network, we first evaluate the limit of the secondary STT, single-hop transport capacity (STC). Then, we investigate STT with secondary receivers randomly located in the field of interest. To provide a comprehensive view of achievable secondary network throughputs, three models regarding the selection of receivers are considered: optimally selected, randomly selected, and the nearest neighbors. Our theoretical analysis are well substantiated by numerical and simulation results.

Index Terms—Cognitive radio, network throughput, outage analysis and transmission capacity

1 INTRODUCTION

The paradox between spectrum scarcity and underutilization strongly propels the development of spectrum sharing systems. In such systems, two or more networks coexist and exploit the shared spectrum more efficiently, provided that no excessive mutual interference is introduced. One prominent example is cognitive radio networks, where secondary (unlicensed) users are allowed to temporarily access spectrum that is not currently used by primary (licensed) users. It is generally preferable that the operation of the secondary network is transparent to the primary network, which requires that the interference incurred by secondary operations be constrained within an acceptable level.

One of the primary concerns on the spectrum sharing systems is network throughput, especially the throughput of the secondary network. Two major measurements have drawn much attention in literature: transport capacity [2], [3], [4] and transmission capacity [5], [6], [7], [8]. The former was defined as the maximum bit meters per second the network can achieve in aggregate [9]. It was shown [9] that the transport capacity of an $n$-node network scales as $\sqrt{n}$. The same scaling is achievable in the secondary network as well, indicated by recent works [2], [3], [4], [10]. Therefore, there is no performance loss for the secondary network in terms of scaling law. Recently the capacity scaling of overlaid networks is considered with a general coding scheme in [11]. Asymptotic analysis on the transport capacity offers insights on the relationship between capacity and network size, but neglects the effects of many important system parameters. Transmission capacity [12], [13] depicts an alternative view, quantifying the maximum spatial density under some outage probability constraint. The outage probability of the secondary network is studied in [5] for spectrum overlay and underlay systems, and in [6] when spectrum sensing is applied. Interestingly if the primary network can tolerate extra outage, the spectrum efficiency of the spectrum sharing systems can potentially improve over the stand-alone ones [7]. Our recent work [14] further derived the sufficient condition for the secondary configuration such that the overall throughput of the overlaid network indeed boosts over the single one. Typically the outage constraints of the primary and secondary network cannot be satisfied simultaneously. The achievable transmission capacity, defined as the product of the spatial density and the actual outage probability, is studied in [8] for the secondary network, and maximized with respect to its transmitter density. Transmission capacity admits quantitative system analysis, but leaves out the consideration (and optimization) of transmission distance, a key parameter for wireless networks.

Endeavoring to obtain a more comprehensive view of network throughput of decentralized overlaid networks, in this paper we propose to study a new metric: single-hop transport throughput (STT), which quantifies the total number of one-hop reliable transmissions in a unit area, weighted by corresponding transmission rates and distances. STT inherits the merits of both traditional transport capacity and transmission capacity, incorporating transmission distance and outage probability into a uniform framework, and quantitatively characterizes the relationship and tradeoff among important network parameters. The analytical results achieved in this work enhance the understanding on network throughput and is insightful for network configuration. Note that our metric is similar in spirit to the random access transport capacity (RTC) recently proposed in [15]. The
difference is that in RTC the transmission distance is predetermined while in STT it is dynamic across the network. On the one hand, we would like to determine the achievable network throughput when links are activated by specific network protocols (such as nearest neighbor routing). On the other hand, it is desired to explore the limiting performance of the network with transmission distance optimized. STT deserves thorough investigation in the spectrum sharing systems, in that single-hop transmissions may be preferred for the secondary network due to its inferior role in the spectrum access; it will also serve as a basis for extension to the multi-hop case, which will be explored in our future work.

We investigate STT of the secondary network in a decentralized setting, subject to outage constraints for both the primary and secondary network, and jointly consider channel randomness and interference, two essential practical factors, in the performance analysis. Our contributions are summarized below:

- Given the settings of the primary network, we first derive the limit of STT, single-hop transport capacity (STC) of the secondary network.
- We further study the STT of the secondary network under the assumption that secondary nodes are randomly distributed according to a Poisson point process (P.P.P) [16]. In such a setting achievable throughput may vary with the way secondary transmitters select their corresponding receivers.
- Three important models are then considered: optimal receivers (OR), random receivers (RR) and nearest neighbors (NN). In the OR model each secondary transmitter chooses an optimal receiver so that the STT is maximized; in the RR model each secondary transmitter randomly chooses its receiver within a transmission radius; and each secondary transmitter chooses the receiver closest to it in the last model. Their performances are compared and some insights are revealed.
- All of our results are in closed-form, which render more insights than the scaling law study widely adopted in literature. Our analysis is also well supported by simulation results.

The remainder of this paper is organized as follows. The system model is given in Section 2, followed by discussions on the feasible density region and the performance limit of the secondary STT in Section 3. Then the STT of the secondary network with randomly (Poisson) distributed nodes is studied in Section 4. And the simulation results are provided in Section 5. Finally the conclusion and future directions are provided in Section 6.

## 2 System Model

In this study, unicast traffic is considered for both the primary and secondary networks, and our focus is on the throughput of the latter, with outage constraints imposed on both. We assume the primary and secondary transmitters are distributed in the same two-dimensional plane, and their positions are modeled as two stationary Poisson point processes; the former is denoted by $\Pi_0^s = \{X_0(i)\} \subset \mathbb{R}^2$ with density $\lambda_{st}$ and the latter by $\Pi_s^t = \{X_s(i)\} \subset \mathbb{R}^2$ with density $\lambda_{sr}$. The locations of potential secondary receivers $Y_s(i)$ are randomly distributed according to a P.P.P $\Pi_s^t$ with density $\lambda_{sr}$. We deliberately consider a general $\lambda_{sr}$ and will explore the effects of different $\lambda_{st}$ and $\lambda_{sr}$ in our study. According to the superposition theorem in [16] the density of secondary nodes (including both transmitters and receivers) is $\lambda_{st} + \lambda_{sr}$. The primary receivers are located within $l_o$ distance from their corresponding transmitters, where $l_o$ is the transmission range of a primary transmitter, considered constant in our analysis; the distribution of the primary receivers is immaterial for our study. No cooperation between the primary and the secondary network is allowed and the common assumptions about their intra-network communications are given below:

- All the primary (secondary) transmitters use the same transmission power $P_p$ ($P_s$) and their power ratio is denoted by $\theta = \frac{P_s}{P_p}$. Concurrent primary and secondary transmissions are simply treated as interference.
- For both networks large-scale path loss and small-scale Rayleigh fading are considered. Particularly the channel power gain for a communication link of length $r$ is given by

$$g(r) = r^{-\alpha}u,$$

where $\alpha > 2$ is the path loss exponent, and $u$ is exponentially distributed with unit mean. The signal to interference and noise ratio (SINR) at a primary receiver $y_{st}$, $r_o$-distance away from its transmitter $x_{st}$, is given by

$$\text{SINR}_{st}(r_o) = \frac{P_p g(r_o)}{N + I_o + I_{so}},$$

where $N$ is the noise power, a constant in our study, and $I_o = \sum_{i \in \Pi_0^s, i \neq o} P_p g(||X_0(i) - y_{st}||)$ ($I_{so} = \sum_{i \in \Pi_0^s, i \neq o} P_p g(||X_0(i) - y_0||)$) is the sum of interference power from concurrent primary (secondary) transmitters, and $|| \cdot ||$ is euclidean norm. The SINR at a secondary receiver, $\text{SINR}_{sr}$, is defined similarly as

$$\text{SINR}_{sr}(r_s) = \frac{P_p g(r_s)}{N + I_s + I_{so}},$$

where $I_s$ ($I_{so}$) is the sum of interference power from concurrent secondary (primary) transmitters to a secondary receiver, $r_s$-distance away from its corresponding transmitter.

- The primary (secondary) transmission is successful if the $\text{SINR}_{st}$ ($\text{SINR}_{sr}$) is no less than a threshold $T_o$ ($T_s$), assumed fixed in our study. The transmission rate is a deterministic function of this threshold $R_o = f(T_o)$ ($R_s = f(T_s)$).

1. One special case of our model is the ALOHA protocol, where a secondary node transmits with probability $\frac{1}{\lambda_{sr} + \lambda_{st}}$.

2. Note that SINR depends on $y_0$, but its distribution (and in particular the outage probability) does not according to our analysis in Section 7.1.
There are outage constraints imposed on primary and secondary transmission links. For the primary network, the constraint is given by:
\[ \Pr(\text{SINR}_o(l_o) < T_o) \leq \epsilon_o, \tag{3} \]
where \( l_o \) is the primary transmission range and \( \epsilon_o(< 1) \) is a predetermined small number. For the secondary network, we consider a similar constraint
\[ \Pr(\text{SINR}_s < T_s) \leq \epsilon_s. \tag{4} \]
Under this constraint, we allow each secondary transmitter the flexibility to choose its corresponding receiver among a set of potential ones, so long as the communication quality is satisfied for each active secondary link, i.e., \( \Pr(\text{SINR}_r(r) < T_s) \leq \epsilon_s \), where the link length \( r \) is variable, and is a key parameter in our study.

For convenience of analysis, it is typically assumed that a primary (secondary) receiver or transmitter is located at the origin, which does not change the statistics of homogenous P.P.P. according to the Slivnyak’s theory [17].

**Remark 1.** The settings of the primary network such as the density and outage probability are assumed fixed without considering the accommodation of the secondary network. In contrast, parameters in the secondary network can be tuned to improve its own performance, provided the primary transmission is not influenced.

In this decentralized framework, the metric transmission capacity has received increasing interest recently, defined as the maximum density of successful transmissions subject to an outage constraint. When the transmitter density of a network is given, a similar concept transmission throughput can be defined. In particular, for a Poisson network with transmitter density \( \lambda \), a typical link length \( r \), a pre-determined SINR threshold \( T \) and transmission rate \( R \), the transmission throughput is defined as:
\[ \bar{C}(\lambda) = R\lambda(1 - \delta(\lambda, r)), \tag{5} \]
where the outage probability \( \delta(\lambda, r) = \Pr(\text{SINR}(r) < T) \). The transmission capacity [12], [13], [15] with outage constraint \( \epsilon \) is defined as \( R\lambda_e(1 - \epsilon) \), where \( \lambda_e \) is the maximal transmitter density satisfying the outage constraint, i.e., \( \delta(\lambda_e, r) = \epsilon \). It can be shown that when \( \epsilon \) is small, the transmission capacity coincides with \( \max_{\lambda} \bar{C}(\lambda) \); but in general, these two metrics are different.

In the study of transmission capacity and its variants, the transmission distance is ignored. In this work, we explore a metric called single-hop transport throughput defined as follows.

**Definition 1.** The single-hop transport throughput of a Poisson network with transmitter density \( \lambda \), a pre-determined SINR threshold \( T \) and transmission rate \( R \), is defined as:
\[ C(\lambda) = R\lambda E_r[r(1 - \delta(\lambda, r))], \tag{6} \]
where the outage probability \( \delta(\lambda, r) = \Pr(\text{SINR}(r) < T) \).

**Remark 2.** The transmission distance explicitly considered in STT is a random variable in general, whose distribution depends on the node distributions (modeled as P.P.P. in our study) and how communication links are formed. In this study, we will assume that \( T \) and \( R \) are fixed and focus our attention on the effect of \( r \) and \( \lambda \). Intuitively, STT indicates that given an SINR threshold \( T \), a transmitter can on average deliver \( R \) bits \( E_r[r(1 - \delta(\lambda, r))] \) distance away in one second.

Communication links can be formed in different manners, depending on how transmitters select their corresponding receivers, which plays an essential role in determining the STT. To present a comprehensive view of achievable secondary STT we are interested in the following selection rules subject to the secondary outage constraint:

1) **Optimal receiver.** The secondary receivers are selected in order to achieve the best performance. Interestingly in the setting of our interest, the optimal receivers selected are the furthest ones from the transmitters.
2) **Random receiver.** The secondary receivers are randomly selected.
3) **Nearest neighbor.** Namely, each secondary transmitter communicates with its nearest neighbor, which is a conservative but easy-to-implement approach.

The upper limit of any achievable STTs is of interest as well.

**Definition 2.** Denote by \( \Pr(r) \) the distribution function of the transmission distance \( r \). The single-hop transport capacity with outage constraint \( \epsilon \) is defined as:
\[ C = \max_{\lambda, \Pr(r)} R\lambda E_r[r(1 - \delta(\lambda, r))], \tag{7} \]
given that \( \delta(\lambda, r) \leq \epsilon \).

**Remark 3.** As the single-hop transport throughput can be viewed as an extension of the transmission throughput, the single-hop transport capacity can be viewed as an extension of the maximal transmission throughput, \( \max_{\lambda} \bar{C}(\lambda) \), which, as we mentioned before, does not necessarily coincide with the conventional definition of transmission capacity in literature. The maximization in Eq. (7) involves the distribution of \( r \), which is in general a challenging task. In this study, it turns out that the optimal distribution of \( r \) is deterministic. For a secondary network, \( \lambda \) and \( r \) should be chosen such that the outage constraint of the primary network is met as well. Further discussion of STC and feasible density regions is given in Section 3.

The notations used in this work are summarized in Table 1.

### 3 Density Region and STC

The purpose of this paper is to explore the single-hop transport throughput of the secondary network given the outage constraints on both the legacy network and itself. We begin by setting the boundaries for this metric. First we determine the feasible density region for the transmitter densities of
operate with low transmission power given limited “white space”. In such circumstances, some low power physical layer techniques such as spread spectrum may help improve the performance of secondary networks.

Unlike the primary outage constraint, the secondary outage constraint alone does not impose pre-determined limitations on the secondary transmitter density. The reason lies in that the operation parameters of the secondary network are subject to our design, which can be adjusted to accommodate both outage constraints. In particular, the secondary outage probability is related to both the secondary transmitter density and transmission distance, both of which are variables in our study.

In the following discussion, we assume all densities of interest \( \lambda_{st}, \lambda_{st} \) are within the feasible density region.

### 3.2 Single-Hop Transport Capacity

To explore the limiting performance of the secondary network, we assume the flexibility to choose an arbitrary distribution for the transmission distance \( r \). It turns out that the optimal distribution is deterministic.

**Lemma 1.** Given a secondary transmitter density \( \lambda_{st} \), assume \( l \) and \( L \) satisfy the following functions respectively:

\[
\cos l^2 + 2B(\lambda_{st})l^2 - 1 = 0, \tag{10}
\]

\[
B(\lambda_{st})L^2 + n_2L^a + \ln(1-e_0) = 0, \tag{11}
\]

where \( n_2 = \frac{T_{st}N}{P} \) and \( B(\lambda) \triangleq K_{st}T_{st}^2/(\lambda^3\theta^3/2 + \lambda) \). The secondary single-hop transport throughput \( C_s(\lambda_{st}) \) achieves the maximum \( C_s(\lambda_{st}) \) by choosing a common transmission distance \( r_s \) for each link,

\[
\begin{align*}
   r_s = \begin{cases} 
   l, & e_2 
\end{cases} \tag{12}
   \end{align*}
\]

In addition both \( l \) and \( L \) decreases with \( \lambda_{st} \).

The proof is given in Section 7.2. Based on the lemma above, given \( \lambda_{st} \), the maximal transport throughput \( \bar{C}(\lambda_{st}) \) is given by:

\[
\bar{C}(\lambda_{st}) = \lambda_{st}r_s\exp(-B(\lambda_{st})r_s^2 - n_2r_s^a), \tag{13}
\]

and the single-hop transport capacity is given by

\[
\bar{C}_s = \max_{\lambda_{st} \in \Phi} \bar{C}(\lambda_{st}).
\]

Typically neither \( l \) nor \( L \) admits closed-form expressions except for some particular \( \alpha \), say \( \alpha = 4 \).

For general \( \alpha \) we can still explore some properties of \( \bar{C}_s \) by directly exploiting Eqs. (10) and (11).

**Theorem 1.** The single-hop transport capacity is

\[
\bar{C}_s = \bar{C}(\lambda_{st}), \tag{14}
\]

where \( \lambda_{st} \) is given in Eq. (9).

The proof is given in Section 7.3.

Since \( \lambda_{st} \) increases with the power ratio \( \theta \), one can always improve the capacity \( \bar{C}_s \) by increasing the power ratio. This implies that in order to achieve high throughput,
secondary transmissions with low power and short transmission range are preferred. As will be shown in Fig. 1 of Section 5, the effect of noise is negligible when sufficiently small, as in most interference-limited scenarios. Thus, we ignore the noise in the following section in order to simplify the computation and better reveal insights. Then, $r_s$ can be simplified as

$$r_s = \left\{ \begin{array}{ll}
  l = 1 / \sqrt{2B(\lambda_{st})}, & \epsilon_s \geq 1 - e^{-1/2}, \\
  L = \sqrt{\frac{\ln(1-\epsilon_s)}{B(\lambda_{st})}}, & \text{otherwise}.
\end{array} \right. \quad (15)$$

And the corresponding maximum single-hop transport throughput $C_S(\lambda_{st})$ is given by

$$\tilde{C}_S(\lambda_{st}) = \left\{ \begin{array}{ll}
  \frac{R_{st}}{\lambda_{st}} e^{-1/2}, & \epsilon_s \geq 1 - e^{-1/2}, \\
  \frac{R_{st}(1-\epsilon_s)}{\sqrt{B(\lambda_{st})}} - \ln(1-\epsilon_s), & \text{otherwise}.
\end{array} \right. \quad (16)$$

Note that the transmission rate $R_s$ increases with the SIR threshold $T_s$, while the optimal transmission distance $r_s$ decreases with it (through $B(\lambda_{st})$). An interesting observation from the equation above is that there exists an optimal $T_s$ such that $C_s(\lambda_{st})$ is maximized. This will be further illustrated in Section 5.

In practice, it is expected that $\epsilon_s$ is smaller than $1 - e^{-1/2} \approx 0.39$. Therefore in the rest of this paper, we focus on this scenario.

Remark 4. It is known from Section 7.2 that $C_s(r_s, \lambda_{st}) = R_{st} r_s (1 - \delta_s(r_s, \lambda_{st}))$ monotonically increases with $r_s \in (0, L)$ when $\epsilon_s < 1 - e^{-1/2}$, and $L$ is the maximum allowable transmission distance satisfying the outage constraint $\epsilon_s$.

4 TRANSPORT THROUGHPUT OF SECONDARY NETWORK

In this section we investigate the single-hop transport throughput of the secondary network defined in Section 2, with the spatial distribution of secondary nodes modeled as a P.P.P. with density $\lambda_{st} + \lambda_{sr}$. The two overlaid networks operate under the primary and secondary outage constraints, i.e., their transmitter densities are within the density region.

Three important settings described at the end of Section 2 are explored, i.e., the OR model where the receivers are selected to maximize the throughput, the RR model where the receivers are uniformly and randomly selected, and the NN model where nearest receivers are chosen. The maximal secondary throughput in Eq. (16) (which is obtained for an arbitrary distribution of receivers and fixed $\lambda_{st}$) serves as a benchmark for our study. Note that in the three models mentioned above, it is possible that multiple secondary transmitters choose the same receiver. It is assumed in such a case that the receiver attempts to decode the packets from each TX separately. For each TX transmission, the receiver considers all the other concurrent transmitting signals as interference and calculates the SINR to check if outage occurs. Therefore, such a scenario has no influence on our study.

4.1 Optimal Receiver

For a secondary transmitter $X_s$, we order the potential secondary receivers $\{Y_i(i)\}$ according to their Euclidean distance $r_i \equiv ||Y_i(i) - X_s||$ to $X_s$, such that $r_i \leq r_j, \forall i < j$. Let

$$k' = \max\{i : L - r_i \geq 0\}.$$ In the OR model, $X_s$ selects $Y_i(k')$ if $r_i \leq L$ (so that there is at least one receiver satisfying the secondary outage constraint), and aborts transmission if $r_i > L$.

Proposition 1. For a secondary network modeled as a P.P.P, the STT is maximized by the OR selection rule.

Proof. According to Remark 4, given the secondary transmitter density $\lambda_{st}$, the throughput monotonically increases over transmission distance $r_s \in (0, L)$. Therefore, the throughput is maximized if each secondary transmitter communicates with the furthest receiver within the range of $L$.

Theorem 2. Given a secondary transmitter density $\lambda_{st}$ and a secondary receiver density $\lambda_{sr}$, the largest achievable single-hop transport throughput is given by:

$$C_s(\lambda_{st}, \lambda_{sr}) = \begin{cases} 
  A_1 \left[ \frac{L_{st}^2 - \sqrt{\pi \alpha \pi L_{st}^2}}{2a} \right], & a > 0, \\
  A_1 \frac{L_{st}^2}{3}, & a = 0, \\
  A_1 \left[ \frac{L_{st}^2}{2a} + \sqrt{\pi \alpha \pi L_{st}^2} \right], & a < 0,
\end{cases} \quad (17)$$

where $A_1 = 2\lambda_{st} R_{st} \lambda_{sr} \pi e^{-\lambda_{sr} \pi L_{st}^2}$, $a = \lambda_{sr} \pi - B(\lambda_{st})$, $B(\lambda_{st})$ is given in Lemma 1, erfi$(x)$ and erfi$(x)$ are the error function and the imaginary error function, given in Table 1.

Proof. Given that there is at least one receiver within the circle with radius $L$, the event $\{r_{kr} > l\}$, where $l \leq L$, is equivalent to the event $\{r_{kr} \leq L\}$. Therefore,

$$\Pr(r_{kr} > l | r_1 \leq L) = \frac{1 - e^{-\lambda_{sr}(l^2 - l^2)}}{1 - e^{-\lambda_{sr} \pi L^2}}.$$
which leads to the conditional pdf of \( r_{1c} \),

\[
f_{r_{1c}|r_{1} \leq L}(l) = \frac{e^{-\lambda_{st} \pi L^2} 2 \lambda_{sr} \pi l}{1 - e^{-\lambda_{st} \pi L^2}} e^{-\lambda_{sr} \pi l^2}.
\]

It follows that

\[
C_s' = \lambda_{st} R_s \Pr(r_1 \leq L) E_{r_{1c}|r_{1} \leq L}(1 - \delta_s(r_{1c}))
\]

\[
= \lambda_{st} R_s \int_0^L \Pr(r_1 \leq L) e^{-B(\lambda_{st})} \int_0^L r \exp(-B(\lambda_{st}) r^2) \frac{2r}{L^2} \frac{dr}{L^2}
\]

where the outage \( \delta_s \) is given in Eq. (29).

\[
\text{Proof.} \ 	ext{The event } \{r_1 > l\} \text{ is equivalent to the event } \{\text{there is no receiver in the circle with radius } l \text{ centered at } X_s\}.
\]

Thus,

\[
\Pr(r_1 > l) = e^{-\lambda_{sr} \pi l^2},
\]

which leads to the conditional pdf of \( r_1 \),

\[
f_{r_1|(l)} = \frac{2 \lambda_{sr} \pi l}{e^{-\lambda_{sr} \pi l^2}}.
\]

Since we only consider the receivers located within \( L \)

\[
C_s' = \lambda_{st} R_s \Pr(r_1 \leq L) (1 - \delta_s(r_1)))
\]

\[
= \lambda_{st} R_s \int_0^L e^{-B(\lambda_{st})} \int_0^L f_{r_1}(l) \frac{dl}{L}.
\]

The theorem then follows after some calculation.

\[
\text{4.2 Random Receiver}
\]

The RR model demonstrates the STT in the average sense, where each secondary receiver randomly chooses a receiver in a range at most \( L \) distance away.

**Theorem 3.** Given a secondary transmitter density \( \lambda_{st} \) and a secondary receiver density \( \lambda_{sr} \), the single-hop transport throughput in the RR model is

\[
C_s' = \frac{A_2}{\sqrt{B(\lambda_{st})}} \left( 1 - \epsilon_s \right) \lambda_{sr} \pi B(\lambda_{st})
\]

where \( A_2 = \frac{\sqrt{\pi} \text{erf}(\sqrt{\ln(\frac{1}{\lambda_{sr}})}) \sqrt{\ln(\frac{1}{\lambda_{sr}})} - 2 \ln(1 - \epsilon_s)}{2 \ln(1 - \epsilon_s)} \) and \( B(\lambda_{st}) \) is given in Lemma 1.

**Proof.** Denote by \( E_k \) the event that there are \( k \) receivers in the circle centered at a secondary transmitter with radius \( L \). Given the number of receivers in the circle, all the receivers are uniformly distributed with probability density function (pdf) \( f(r) = \frac{2}{\pi r} \) according to the Poisson property. Then we have

\[
E[r_1 - \delta_s(r_1)] = E[E(r_1 - \delta_s(r_1) | E_k)]
\]

\[
= \sum_{k=1}^{\infty} \left( \frac{\lambda_{sr} \pi L^2}{k!} \int_0^L r \exp(-B(\lambda_{st}) r^2) \frac{2r}{L^2} \frac{dr}{L^2} \right)
\]

\[
= \left( 1 - \frac{1}{e^{\lambda_{sr} \pi L^2}} \right) \frac{\sqrt{\pi} \text{erf}(\sqrt{B(\lambda_{st})})}{2BL(\lambda_{st})} - \frac{e^{-B(\lambda_{st}) L^2} B(\lambda_{st})}{B(\lambda_{st}) L}.
\]

The theorem follows after substituting \( L \) in Eq. (12) to the equation above.

\[
\text{4.3 Nearest Neighbor}
\]

Nearest neighbors model is one of the popular models in analysis of wireless networks, especially in the study of network connectivity ([19] and reference there). In this section we consider secondary transmitters communicate with their nearest receivers satisfying the outage constraints. For the ordered receivers introduced in Section 4.1, the receiver chosen by \( X_s \) in this model is \( Y_s(1) \) if \( r_1 \leq L \); otherwise no receiver is chosen since all the receivers are outside of the allowable transmission range.

**Theorem 4.** Given a secondary transmitter density \( \lambda_{st} \) and a secondary receiver density \( \lambda_{sr} \), the STT in the NN model is given by

\[
C_s''(\lambda_{st}, \lambda_{sr}) = A_1 \left( \frac{L e^{-B(\lambda_{st}) L^2}}{2b} + \frac{\sqrt{\pi} \text{erf}(\sqrt{b} L)}{4e^{-\lambda_{sr} \pi L^2} b^{3/2}} \right),
\]

where \( b = B(\lambda_{st}) + \lambda_{sr} \pi, B(\lambda_{st}) \) is given in Lemma 1, and \( A_1 \) is given in Theorem 2.

**Proof.** Consider a circle \( \pi(\lambda_{sr}) \) of radius \( a \) centered at the origin and denote by \( N_a \) the number of receivers in \( \pi(\lambda_{sr}) \) connected to the origin, then we have

\[
E(N_a) = \sum_{l=1}^{\infty} I_l | N_a = l \]

\[
= \sum_{l=1}^{\infty} \sum_{k=1}^{l} I_l | N_a = k \]

\[
= \sum_{k=1}^{\infty} I_k | N_a = k \]

where \( N_a = \Pi^*(\pi(\lambda_{sr})) \) is the number of receivers in \( \pi(\lambda_{sr}) \) and \( I_l \) is the indicator function such that \( I_l = 1 \) if receiver \( l \) is connected to the origin.

The probability density function that a secondary receiver is at a distance \( r < a \) from the origin is \( f_r(r) = \frac{2r}{\pi a^2} \). Therefore,

\[
E \left( \sum_{l=1}^{\infty} I_l | N_a = k \right) = kE(I_l) = k \int_0^a (1 - \delta_s) f_r(r) dr,
\]

where \( \delta_s \) is given in Eq. (29).

Taking Eq. (22) into Eq. (21) and noticing the fact that \( \sum_{k=1}^{\infty} \frac{\lambda_{sr} \pi a^2 (1 - e^{-\lambda_{sr} \pi a^2})}{(k-1)!} = 1 \), we get
Thus, 
\[
k = \lim_{a \to \infty} E(N_a) = \frac{\lambda_{s\epsilon} \pi}{B(\lambda_{st})}.
\]

Define \( \beta = \frac{\lambda_{s\epsilon}}{\lambda_{st}} \), the density ratio between the secondary receivers and transmitters. Given \( \beta \)
\[
k = \frac{\beta \pi}{K_a T_s^{2/3} \lambda_{st}^{1/3} + 1} = \frac{\pi \beta \Delta \lambda_{ot}}{K_a T_s^{2/3} \lambda_{ot}},
\]
where the inequality is due to the fact that \( \lambda_{st} \leq \lambda_{ot} \). This indicates that in a \( p \)-Persistent Aloha system with probability \( p = \frac{1}{N} \) such that a secondary node becomes a transmitter, the average node degree is upper bounded with a constant.

Substituting the average node degree \( k = \frac{\lambda_{s\epsilon}}{B(\lambda_{st})} \) into Eq. (17) \( \alpha > 0 \) case, (18) and (19), we have
\[
C_s^o(\lambda_{st}, k) = Q_s(k) \frac{R_s \lambda_{st}}{\sqrt{B(\lambda_{st})}}, \quad (k > 1),
\]
\[
C_s^e(\lambda_{st}, k) = Q_e(k) \frac{R_s \lambda_{st}}{\sqrt{B(\lambda_{st})}},
\]
\[
C_s^o(\lambda_{st}, k) = Q_o(k) \frac{R_s \lambda_{st}}{\sqrt{B(\lambda_{st})}},
\]
where
\[
Q_s(k) = \frac{k \sqrt{\ln(1 - \epsilon_s)}}{k - 1} - \frac{k \sqrt{\ln(1 - \epsilon_s)}}{2(k - 1)^{3/2}(1 - \epsilon_s)^{5/2}},
\]
\[
Q_e(k) = A_2 (1 - (1 - \epsilon_s)^k),
\]
\[
Q_o(k) = \frac{k \sqrt{\ln(1 - \epsilon_s)}}{2(k - 1)^{1/2}} - \frac{k \sqrt{\ln(1 - \epsilon_s)}}{2(k - 1)^{1/2}} (1 - \epsilon_s)^{1+k} + \frac{k \sqrt{\ln(1 - \epsilon_s)}}{2(k - 1)^{1/2}} (1 - \epsilon_s)^{1+k}.
\]

All the above results, as well as the maximal throughput derived in Eq. (16), take the same form except for the coefficients \( Q_i, i = o, r, n \). This similarity provides a common ground for our following discussion.

**Corollary 1.** \( C_i^o(\lambda_{st}) = C_i^o(\lambda_{ot}) = \Theta(\sqrt{\lambda_{ot}}) \), where \( i = o, r, n \) and \( C_i \) is given in Eq. (16).

**Proof.** Since \( C_i^o \) and \( C_i^e \) only differ in a constant value in this scale law study, we focus on \( C_i^o(\lambda_{st}) \) below.

Since the secondary transmitter density \( \lambda_{st} \) increases with the power ratio \( \theta \), according to Eq. (9), we have
\[
\lim_{\lambda_{st} \to \infty} \frac{C_s(\lambda_{st})}{\lambda_{st}} = (1 - \epsilon_s) \frac{\sqrt{-\ln(1 - \epsilon_s)}}{\sqrt{B(\lambda_{st})}}.
\]

Given the setting of the primary network this ratio is a positive constant. Thus the proof is completed.

**Remark 5.** It is shown in [20], [21] that, in a single \( n \)-node dense network, where the network size is allowed to grow with the network density in a fixed area, the sum throughput scales with \( \Theta(\sqrt{n}) \). The upper bound of the sum throughput is derived in the one-shot (equivalently one-hop) setting [20]. It is expected that the STC and STT in the secondary network may degrade from that in the stand-alone network. However, our results show that the same scaling law is maintained. In addition, the critical constant term is usually unavailable in the scaling law study, while it is clearly presented in our expressions. In addition, we can see that with larger spectrum opportunity \( \Delta \lambda_{ot} \), the secondary throughput increases faster.

**Corollary 2.**

1) Given secondary transmitter density \( \lambda_{st} \), \( C_o^o \) and \( C_o^e \) in the OR and RR model increase with the secondary receiver density \( \lambda_{sr} \), or equivalently \( k \), and
\[
\lim_{k \to \infty} C_o^o(\lambda_{st}, k) = \overline{C}_o(\lambda_{st}),
\]
\[
(1 - \epsilon_s) \sqrt{-\ln(1 - \epsilon_s)} \frac{R_s \lambda_{st}}{\sqrt{B(\lambda_{st})}}.
\]
\[
\lim_{k \to \infty} C_o^e(\lambda_{st}, k) = A_2 \frac{R_s \lambda_{st}}{\sqrt{B(\lambda_{st})}},
\]
where \( \overline{C}_o(\lambda_{st}) \) is given in Eq. (16) and \( A_2 \) is given in Theorem 3.

2) Given \( \lambda_{st} \), \( C_o^o \) in the NN model is maximized at \( k \approx \frac{2}{\epsilon_s} \) for small \( \epsilon_s \).

3) Given \( \beta, C_o^o \) and \( C_o^e \) increases with \( \lambda_{st} \) and achieve the maximum at \( \lambda_{st} = \lambda_{ot} \).

**Proof.** Property 1: It is straightforward to see that \( Q_s(k) \) and \( Q_e(k) \) increases with \( k \) and the convergence is achieved after some calculation.

Property 2: The existence of an optimal \( k \) such that \( C_o^o \) is maximized is due to the fact that \( \lim_{k \to 0} C_o^o(\lambda_{st}, k) = \lim_{k \to \infty} C_o^o(\lambda_{st}, k) = 0 \). To get the optimal \( k \) we solve the equation \( \frac{dQ_s(k)}{dk} = 0 \).

\[
\frac{dQ_s(k)}{dk} \approx \frac{-\ln(1 - \epsilon_s)}{1 + k} \ln(1 - \epsilon_s) \approx \frac{\epsilon_s^2}{2} \left( -k^2 + \frac{2 \epsilon_s}{k} \right),
\]
where the approximation is made due to the fact that \( \text{erf}(\sqrt{(1 + k) \ln(1 - \epsilon_s)}) \approx 2 \sqrt{\pi} \sqrt{(1 + k) \ln(1 - \epsilon_s)} \).
and \((1 - \epsilon_s)^{1+k} \approx 1 - \epsilon_s(1 + k) + \epsilon_s^2 k(1 + k)/2\) for small \(\epsilon_s\). The optimal \(k \approx \frac{2}{\epsilon_s}\).

Property 3): Since \(Q_o(k)\) and \(Q_r(k)\) increase with \(k\), which increases with \(\lambda_{st}\) given \(\beta\) (c.f. Eq. (24)), they also increase with \(\lambda_{st}\) and achieves the maximum at \(\lambda_{st} = \bar{\lambda}_{st}\).

**Remark 6.** Intuitively, Property 1) indicates that the capacity discussed in the last section is achieved when the density of receivers in the OR model goes to infinity (so that a secondary receiver can be found at the optimal distance), and the RR model suffers performance loss by a constant factor. In addition, the NN model achieves the optimal value at a finite \(k\) value, whereas the other two achieve the optimum when \(k\) goes to infinity. Property 3) reveals that in a \(p\)-Persistent Aloha system, the secondary throughput in the OR and RR model increases with the secondary density \(\lambda_{st}\), as well as the total secondary density \(\lambda_{st} + \lambda_{sr}\) (since \(\beta\) is fixed). Note that if the receiver density is fixed, the throughput does not necessarily increase with the transmitter density, as can be seen from the expressions in Eqs. (17) and (18).

### 5 Numerical and Simulation Results

In this section, the above analytical results about STT and STC are evaluated numerically to further reveal their dependence on system parameters. We also provide the corresponding simulation results to verify the correctness of our analysis. In the simulation, the primary and secondary nodes are randomly and uniformly distributed in a square \([0,1.000]^2\) with their respective densities. The system parameters are set as follows, unless otherwise noted: \(T_s = 3\), \(T_r = 2\), \(l_o = 6\), \(\alpha = 4\), \(\epsilon_s = \epsilon_o = 0.1\), \(P_o = 2\), \(P_s = 1\), \(\lambda_{st} = \bar{\lambda}_{st}\) and \(\lambda_{sr} = \frac{1}{5}\bar{\lambda}_{st}\). As can be seen from the figures described below (where the throughputs are normalized by the (constant) transmission rate \(R\)), our analytical results (solid lines) agree well with the simulation results (denoted by markers). In the following, we give further discussions on these results and offer some insights.

Fig. 1 depicts the maximal transport throughput with respect to the secondary transmitter density when noise is explicitly considered. When \(\alpha = 4\), it’s easy to check that

\[
T_s = \begin{cases} 
L = \sqrt{\frac{-B(\lambda_{st}) + \sqrt{B^2(\lambda_{st}) + 4n}}{4n}}, & \epsilon_s \geq \epsilon_1, \\
L = \sqrt{\frac{-B(\lambda_{st}) + \sqrt{B^2(\lambda_{st}) - 4n\ln(1 - \epsilon)}}{2n}}, & \text{otherwise},
\end{cases}
\]

where \(\epsilon_1 = 1 - \exp(-\frac{1}{4} + \frac{B^2(\lambda_{st}) - B(\lambda_{st})B(\lambda_{sr})}{8n}\frac{B(\lambda_{sr})}{B(\lambda_{st}) + 4n})\).

Then, the maximal secondary transport throughput can be numerically evaluated through Eq. (13).

We can see from Fig. 1 that, the maximal STT increases with the transmitter density and achieves STC at the maximal permissible transmitter density, which is the end point of each curve and agrees well with our analysis in Theorem 1. As expected, both the STT and STC decrease with the increasing noise power. However when \(N < -40\) dBm, the influence of noise is negligible.

Fig. 2 reveals the relation between the maximal STT and the SINR threshold \(T_s\) (where \(\lambda_{st}\) is chosen as \(\frac{\lambda_{st}}{5}\)) for better illustration. As can be seen, there exists an optimal \(T_s\) such that the (maximal) STT is maximized. In addition, it is observed that the maximal STT increases rapidly when \(T_s\) is small, while decreases very slowly after passing its peak value. This indicates that in practice we only need to ensure that the SINR threshold exceeds a certain critical number to avoid significant performance degradation.

The STTs of the three models (Eq. (17) for the OR model, (18) for the RR model and (19) for the NN model) discussed in Section 4 are depicted in Figs. 3, 4, 5 and 6. In Figs. 3 and 4, the evolution of STTs with the secondary receiver density is revealed for \(\epsilon_s = 0.1\) and \(\epsilon_s = 0.3\), respectively, and compared with their upper bound given in Eq. (16). We observe from both figures that, in the low receiver density region, three models perform almost the same. However, with the receiver density increased, their throughputs start to deviate. The throughputs of both the OR and RR model increase with \(\lambda_{sr}\), and the former approaches the upper bound (16). In contrast, the throughput of the NN model achieves the maximum at \(\lambda_{sr} \approx \frac{2B(\lambda_{sr})}{\epsilon_s R}\), and decreases afterwards. These findings coincide with our analysis in Corollary 2, and admit an intuitive interpretation based on
our observations in the simulation: with a low receiver density the receivers chosen by a transmitter in these models often coincide so that their performance is indiscernible; with more receivers available, the transmitters more likely select the receivers close to the optimal positions in the OR and RR model; however, the distance between a transmitter and its closest receiver becomes smaller as the receiver density increases, which eventually dominates the throughput performance in the NN model. Another interesting observation is that with a low outage probability imposed on the secondary network (Fig. 3), the throughput of the OR model approaches the upper bound rather slowly with the increase of the secondary receiver density; its converging speed is substantially faster with a higher outage probability (Fig. 4). The reason behind this phenomenon is that a higher outage probability indicates a larger maximum allowable transmission distance (c.f. (15) and Remark 4), and in turn more potential receivers to select for a given receiver density.

Fig. 5 describes the throughputs for different transmitter densities given the density ratio $\beta = 100$. The throughputs increase with the transmitter density and scales as $\sqrt{\lambda_{st}}$, which substantiates our analysis in Corollary 1.

6 CONCLUSIONS AND FUTURE WORK

We have made some quantitative study on the throughput of the secondary network in spectrum sharing systems subject to the outage constraints for both the legacy network and the secondary network, aiming at revealing the relationship and tradeoff among key system parameters and providing insights into system design and optimization.

One extension of our study in the future is to explore the multi-hop transport throughput. Some pioneer work in this direction can be found in [15], [18], [22]. Note that if the distance $S$ between a pair of secondary nodes is fixed, the corresponding unicast transport throughput is maximized when $S/l$ equidistant relays can be selected on the path, where $l$ is given in Lemma 1. This conclusion coincides with that in [15]. Another extension is to explore the network transport throughput when MIMO technology is incorporated. In our current study, concurrent transmissions are simply treated as interference and the system performance is interference limited. MIMO technique could potentially enhance the performance, and [21] may provide some guidelines in this direction.

7 DETAILED PROOFS

7.1 Derivation of Outage Probability

Overlaid with the secondary network, the probability of a successful primary transmission, $\Pr(SINR_o(l_o) \geq T_o)$, is given by:

4. In a multi-hop transmission the focus is on the end-to-end delay, so there is a tradeoff between outage probability and hop distance. It turns out that longer hop distance ($l$ versus $L$) is favored.
where $n_o = \frac{N}{T_o}$, $I = I_o + I_{so}$, with pdf $f_I$, and $\psi_I(i)$ is the Laplace transform of $f_I$. Due to the independence of $I_{so}$ and $I_{so}$, which are modeled as the shot noise,

$$\psi_I P_r = \psi_{I_{so}} \frac{P_{r_{so}}}{P_o} \psi_{I_{so}} \frac{P_{r_{so}}}{P_o} \exp \left\{ -K_o T_o^{2/a} \left( \lambda_d \frac{1}{6^2/a} + \lambda_{sd} \right) \right\},$$

(28)

where $\psi_I(i) = \exp \left\{ -K_o \lambda_{sd} (P_{r_{so}})^{2/a} \right\}$, $\psi_{I_{so}}(i) = \exp \left\{ -K_o \lambda_{sd} (P_{r_{so}})^{2/a} \right\}$, where $K_o$ is defined after (8). Eq. (9) is obtained by letting $\Pr(SINR_{I_s} \leq T_o) \geq 0$.

Following the same line above, the outage probability of the secondary network can be calculated as follows:

$$\delta_s(r, \lambda_d) = \Pr \left( \frac{P_{s_{w}} r^{-a}}{N + I_o + I_{so}} \leq T_s \right)$$

$$= 1 - \exp \left\{ -K_o r^{2/a} \left( \lambda_d \frac{1}{6^2/a} + \lambda_{sd} \right) \right\},$$

(29)

where $n_o = \frac{N}{T_o}$, $r$ is the transmission distance between a secondary transmitter and its corresponding receiver.

7.2 Proof of Lemma 1

From the definition of transport throughput in Eq. (6),

$$C_s(\lambda_d) = R_s \lambda_d E_c[r(1 - \delta_s(r, \lambda_d))],$$

$$\leq \lambda_d C_s(r, \lambda_d) \leq \overline{C}_s(\lambda_d),$$

subject to the outage constraint $\delta_s(r, \lambda_d) \leq \epsilon_s$, where $C_s(r, \lambda_d) = R_s \lambda_d r_s (1 - \delta_s(r, \lambda_d))$ and $\delta_s(r, \lambda_d) = 1 - e^{-\lambda_d r_s^{1-\epsilon_s}}$. is given in (29).

Taking derivative of $C_s(r, \lambda_d)$ w.r.t $r_s$ we have

$$\frac{dC_s(r, \lambda_d)}{dr_s} = \frac{R_s \lambda_d (1 - 2B(\lambda_d) r_s^2 - an_s r_s^{a+1})}{\exp \left\{ B(\lambda_d) r_s^2 + an_s r_s^{a+1} \right\}}.$$

Note that $f(r_s) = 1 - 2B(\lambda_d) r_s^2 + an_s r_s^{a+1}$ is monotonically decreasing. Regardless of the outage constraint $C_s(r, \lambda_d)$ achieves maximum when $f(r_s) = 0$, i.e., $r_s = l$. The outage probability at $r_s = l$ is $1 - e^{-\frac{1}{2} \left( \frac{a+1}{a} \right) a r_s^a}$ and $C_s(r, \lambda_d)$ (and outage) monotonically increases with $r_s$ when $r_s \in (0, l)$. Thus, the throughput is maximized at $r_s = l$ when $\epsilon_s \geq 1 - e^{-\frac{1}{2} \left( \frac{a+1}{a} \right) a r_s^a}$. Otherwise, the throughput is maximized when the outage constraint $\epsilon_s$ is achieved, i.e., $\delta_s(r, \lambda_d) = \epsilon_s$ and the corresponding transmission distance is given by $r_s = L$.

Taking derivative of both sides of Eq. (10) w.r.t $\lambda_{sd}$, we get

$$\frac{dl}{d\lambda_{sd}} = -\frac{2K_o T_o^{2/a^2}}{\alpha^2 n_s a^{-1} + 4B(\lambda_d)} < 0.$$

Thus, $l$ decreases with $\lambda_{sd}$. Similarly $L$ decreases with $\lambda_{sd}$ as well.

7.3 Proof of Theorem 1

It suffices to show that $C_s(\lambda_d)$ increases w.r.t $\lambda_d$. Note that $r_s$ is also a function of $\lambda_{sd}$.

When $r_s = l$ the first derivative of $C_s(\lambda_d)$ is given by:

$$\frac{dC_s(\lambda_d)}{d\lambda_{sd}} \exp \left\{ B(\lambda_d) r_s^2 + an_s r_s^{a+1} \right\}$$

$$= r_s + \lambda_d \frac{dr_s}{d\lambda_{sd}} + \lambda_d \left( -2B(\lambda_d) r_s^2 \frac{dr_s}{d\lambda_{sd}} - K_o r_s^{2/a} - an_s \frac{dr_s}{d\lambda_{sd}} \right)$$

$$= r_s + \lambda_d \frac{dr_s}{d\lambda_{sd}} + \lambda_d \left( -\frac{dr_s}{d\lambda_{sd}} - K_o r_s^{2/a} \right)$$

where the second equality follows from Eq. (10). Also from Eq. (10), $r_s \leq \frac{1}{2B(\lambda_d)}$. Then,

$$\frac{dC_s(\lambda_d)}{d\lambda_{sd}} = \frac{r_s \left( 1 + \lambda_d K_o r_s^{2/a} \right)}{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}} > 0.$$

When $r_s = L$ we first take the derivative of both sides of Eq. (11) w.r.t $\lambda_{sd}$,

$$2B(\lambda_d) r_s^2 \frac{dr_s}{d\lambda_{sd}} + K_o r_s^{2/a} \frac{dr_s}{d\lambda_{sd}} + an_s \frac{dr_s}{d\lambda_{sd}} = 0$$

$$\Rightarrow \frac{dr_s}{d\lambda_{sd}} = -\frac{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}}{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}}.$$

Then,

$$\frac{dC_s(\lambda_d)}{d\lambda_{sd}} \exp \left\{ B(\lambda_d) r_s^2 + an_s r_s^{a+1} \right\}$$

$$= r_s + \lambda_d \frac{dr_s}{d\lambda_{sd}}$$

$$= 2B(\lambda_d) r_s^2 + an_s r_s^{a+1} - \lambda_d K_o r_s^{2/a} r_s^3 + \frac{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}}{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}}$$

$$= \frac{r_s^3 (2B(\lambda_d) - \lambda_d K_o r_s^{2/a})}{2B(\lambda_d) r_s^2 + an_s r_s^{a+1}},$$

where the first equality is obtained according to Eq. (30) and the second equality is achieved after plugging in Eq. (31). This completes the proof.

ACKNOWLEDGMENTS

This work was supported in part by the US National Science Foundation under Grants ECCS-1002258, ECCS-1307949 and EARS-1444009. It was partly presented at the 30th IEEE International Conference on Computer Communications (IEEE INFOCOM 2011), Shanghai, China, April, 2011 [1]. Huaiyu Dai is the corresponding author.
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