Analysis on the Diversity-Multiplexing Tradeoff for Ordered MIMO SIC Receivers

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Abstract

The diversity-multiplexing tradeoff for multiple-input multiple-output (MIMO) point-to-point channels and multiple access channels are first proposed and studied by Zheng and Tse recently. While the optimal tradeoff curves for MIMO channels have been explicitly explored, those corresponding to some practical MIMO schemes are still open. One such important problem is the diversity-multiplexing tradeoff for a V-BLAST type system employing ordered successive interference cancellation (SIC) receivers with zero forcing (ZF) or minimum mean square error (MMSE) processing at each stage. In this paper, we take a novel geometrical approach and rigorously verify that under general settings, the optimal ordering rule for a V-BLAST SIC receiver will not improve its performance regarding diversity-multiplexing tradeoff in point-to-point channels. The same geometrical tool is then applied to MIMO spatial-division multiple access channels, leading to some first results in this area. Particularly, we reveal that when the data streams of different users are transmitted with fixed rate (i.e., zero spatial multiplexing gain), the diversity order is not improved by user ordering.

Key Words:

Diversity-Multiplexing Tradeoff, Multiple Access Channels, Ordered SIC Receiver, SDMA, V-BLAST

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I. Introduction

It is well known that multiple-input multiple-output (MIMO) fading channels can be explored to provide either spatial multiplexing gain or diversity gain. However, these two gains typically compete with each other, and the tradeoff between them is represented by the diversity-multiplexing tradeoff curve proposed in the pioneering work [6]. In [7], the study is extended to multiple access channels, where the fundamental tradeoff among diversity gain, multiplexing gain and multiple access gain are effectively characterized. The tradeoff discussions in [6] and [7] mainly deal with the information-theoretic performance limits, where optimal joint encoding and decoding are assumed. There is also some study on sub-optimal (and practical) encoding and decoding schemes, for example, the V-BLAST architecture [1]-[3], where different substreams are separately encoded and transmitted without any transmitter side channel state information (CSI) [6]. The discussion on V-BLAST is naturally extended to spatial-division multiple access (SDMA) channels in [7], where data streams from independent transmitters equipped with multiple antennas are separately encoded, but can be jointly detected (through multiuser detectors—MUD).

This paper mainly exploits a geometrical analysis to investigate the impact of optimal ordering on the performance of a V-BLAST type system in point-to-point and SDMA channels, where successive interference cancellation (SIC) detectors are employed, and zero forcing (ZF) or minimum mean-square-error (MMSE) linear processing is applied at each stage for residue interference cancellation. In [6][7], the diversity-multiplexing tradeoff in these schemes with fixed ordering is accurately quantified. However, when ordering is involved, only some loose performance upper bounds are provided (using genie-aided approaches). The difficulty mainly lies in that, the explicit distributions of the ordered channel gains (or capacities in the SDMA case) are intractable; it is in general a problem of order statistics among inter-dependent random variables, an under-developed topic itself [11]. Recently, some analysis for the ordered SIC receivers with fixed data rate (i.e. the spatial multiplexing gain $r = 0$) is given for some simplified scenarios, together with some conjectures from numerical results. In particular, for point-to-point channels, in [4][5][8] the diversity order for a two-layer V-BLAST scheme with an ordered SIC receiver is rigorously shown to be equal to that with fixed ordering; while in [4][9] numerical results are provided to show that, the diversity order of SIC receivers in a V-BLAST scheme with more than two layers is not increased by the ordering rule proposed in [1]. Except [7], there is little work on the effect of user ordering for SDMA systems with SIC detectors.

In this paper, using a geometrical approach, we first rigorously prove that ordering will not improve the performance of V-BLAST SIC receivers in point-to-point MIMO channels, with respect to (w.r.t.) the diversity-multiplexing tradeoff, thus verifying the conjectures in [4][9]. Similar results w.r.t. the diversity order (when $r = 0$) are also given in the independent work [16]. However, the geometrical approach introduced in this paper is novel and essentially different from the techniques in [16]. This geometrical tool also allows us to extend the study to MIMO SDMA SIC receivers, leading to tighter bounds than those provided in [7]. Particularly, we reveal that when the data streams of different users are transmitted with fixed rate, the diversity order is not improved by user ordering. To the best of our knowledge, these are among the first results in this area.

The rest of the paper is organized as follows. The system model and problem formulation are provided in Section II. Our result on the diversity-multiplexing tradeoff for ordered V-BLAST SIC receivers in point-to-point MIMO channels is presented in Section III, and is extended to MIMO SDMA channels in Section IV. Finally Section V contains some concluding remarks.

II. System Model and Problem Formulation

A. Point-to-Point MIMO Channels

We consider a frequency non-selective block Rayleigh fading channel model, for which a point-to-point system can be expressed as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} \,, \tag{1}$$

where the $N_R \times t$ matrix $\mathbf{y}$ is the received signal block; the $N_T \times t$ matrix $\mathbf{s}$ is the transmitted signal block, each row (layer) representing one separately encoded data stream with power constraint ρ_0 / N_T; $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{N_T}]$ is an $N_R \times N_T$ channel matrix with the constraint $N_R \geq N_T$; and $\mathbf{n}$ is the background noise matrix of size $N_R \times t$. The entries of $\mathbf{H}$ and $\mathbf{n}$ are assumed independent and identically distributed (i.i.d.) complex Gaussian with zero mean and unit variance. Because we focus on the non-ergodic quasi-static fading scenario, the coding length t of each layer is actually immaterial in our study on the diversity orders. As is known, layered one-dimensional coding over block fading channels can only bring coding gain but not the diversity gain. In the following analysis, we will use notation "^" to indicate the normalized (unit norm) counterpart of a vector, e.g., $\hat{\mathbf{h}}_k = \mathbf{h}_k / \|\mathbf{h}_k\|$, and use $\hat{\mathbf{H}}$ for $[\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \cdots, \hat{\mathbf{h}}_{N_T}]$.

Given the above channel model, first consider a ZF-SIC receiver in a V-BLAST system. At the first decoded layer[1], a spatial equalizer $\mathbf{g}_1 = \left[\mathbf{H}^{\dagger}\right]_{(1)*}$ [2]is applied to the received signal $\mathbf{y}$ to obtain an estimate of the transmitted symbol:

$$\hat{s}_{(1)} = \mathbf{g}_1 \mathbf{y} = s_{(1)} + \mathbf{g}_1 \mathbf{n} \,. \tag{2}$$

The instantaneous channel capacity and error performance of this layer are determined by the corresponding post-processing signal-to-noise ratio (SNR), given as

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[1] Due to possible ordering, we let $(l) \in [1, N_T]$ be the transmit antenna index corresponding to the *l*th decoded layer.

[2] $\mathbf{A}^{\dagger}$ denotes the pseudo-inverse of $\mathbf{A}$ and $[\mathbf{A}]_{(i)*}$ is the (i)-*th* row of matrix $\mathbf{A}$.

$$\rho_1^{(ZF-SIC)} = \left(\frac{\rho_0}{N_T}\right) \Big/ \|\mathbf{g}_1\|^2 = \left(\frac{\rho_0}{N_T}\right) \Big/ \left[\mathbf{H}^H\mathbf{H}\right]^{-1}_{(1)(1)} = \frac{\rho_0}{N_T} R_{(1),span\{(2),(3),\ldots,(N_T)\}}, \quad (3)$$

where $R_{(1),span\{(2),(3)\ldots(N_T)\}} \triangleq 1/\left[\mathbf{H}^H\mathbf{H}\right]^{-1}_{(1)(1)}$ is equivalent to the square of the projection height[3] from $\mathbf{h}_{(1)}$ to the space spanned by $\mathbf{h}_{(2)}\ldots\mathbf{h}_{(N_T)}$, denoted as $span\{(2),(3)\ldots(N_T)\}$. Similarly, for the lth decoded layer, assuming perfect decision feedback, the spatial equalizer $\mathbf{g}_l$ is designed to null out the interference from the yet-to-be detected layers, and

$$\rho_l^{(ZF-SIC)} = \frac{\rho_0}{N_T} R_{(l),span\{(l+1),\cdots,(N_T)\}}. \quad (4)$$

For a MMSE-SIC receiver, the spatial equalizer for the first decoded layer is given by $\mathbf{g}_1 = \left[\left(\mathbf{H}^H\mathbf{H} + N_T/\rho_0\mathbf{I}\right)^{-1}\mathbf{H}^H\right]_{(1)*}$, and the corresponding post-processing SNR is:

$$\rho_1^{(MMSE-SIC)} = \frac{\rho_0}{N_T[\mathbf{H}^H\mathbf{H} + N_T/\rho_0\mathbf{I}]^{-1}_{(1)(1)}} - 1 = \frac{\rho_0\gamma_1^{MMSE-SIC}}{N_T} - 1, \quad (5)$$

where $\gamma_1^{(MMSE-SIC)} \triangleq \frac{1}{[\mathbf{H}^H\mathbf{H} + N_T/\rho_0\mathbf{I}]^{-1}_{(1)(1)}}$. We can similarly define $\rho_l^{(MMSE-SIC)}$ and $\gamma_l^{(MMSE-SIC)}$ for $l = 2\ldots N_T$.

Before proceed, we first introduce some notations. The diversity order of a V-BLAST system with SIC receivers is defined as the slope of the average joint error probability $\mathrm{P}_e(\rho_0)$[4] in log-scale at the high SNR regime:

$$d = -\lim_{\rho_0\to\infty}\frac{\log \mathrm{P}_e(\rho_0)}{\log(\rho_0)} = \lim_{x\to 0}\frac{\log \mathrm{P}_e(1/x)}{\log x}. \quad (6)$$

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[3] Projection height refers to the norm of the error vector, i.e., the difference between a vector and its projection onto a subspace.

[4] By joint error probability, we mean an error is declared if any layer is decoded unsuccessfully. For each layer, assuming coding over a single block with constant fading, this is the true error probability of a code averaged over the transmitted codewords, channel fading, and additive noise.

Similar to [6], we use $f_\rho(\rho) \doteq \rho^{-b}$, $b > 0$ to represent $\lim_{\rho\to\infty} \frac{\log f_\rho(\rho)}{\log \rho} = -b$, and $f_\rho(\rho) \dot{\leq} \rho^{-b}$ for $\lim_{\rho\to\infty} \frac{\log f_\rho(\rho)}{\log \rho} \leq -b$ or larger diversity order, and $\dot{\geq}$ is similarly defined[5]. Multiplexing gain r is defined as $r = \lim_{\rho_0\to\infty} \frac{R(\rho_0)}{\log \rho_0}$, where $R(\rho_0)$ stands for the data rate (bps/Hz). Corresponding to each $r \neq 0$, a family of codes $\{\varsigma(\rho_0)\}$ over a block length shorter than fading coherence time is employed, one at each SNR level ρ_0 achieving the corresponding rate $R(\rho_0)$.

The outage probability of the V-BLAST system with a SIC receiver is defined as [6]

$$P_{out}(\rho_0) = \Pr\left(C(\rho_0) \leq R(\rho_0)\right),$$

where $C(\rho_0)$ is the sum of the Shannon capacities of all N_T subchannels. Assuming equal-rate allocation without loss of generality, the outage probability of the *l*th decoded layer (with perfect feedback) can be expressed as (where ρ_l stand for $\rho_l^{(ZF-SIC)}$ or $\rho_l^{(MMSE-SIC)}$)

$$P_{out}^{(l)}(\rho_0) = \Pr\left(\log\left(1+\rho_l\right) \leq \frac{R(\rho_0)}{N_T}\right) \quad . \tag{7}$$

Furthermore, it was shown in [6][10] that in non-ergodic scenarios the error probability, either for the whole system (P_e), or for each layer ($\{P_e^{(l)}\}$), is dominated by the outage probability, i.e.,

$$P_e(\rho_0) \doteq P_{out}(\rho_0) \text{ and } P_e^{(l)}(\rho_0) \doteq P_{out}^{(l)}(\rho_0). \tag{8}$$

With these definitions, the following lemma allows us to focus the study on ZF-SIC receivers.

Lemma I*: For any ordering rule, ZF-SIC and MMSE-SIC receivers in V-BLAST systems achieve the same diversity order.*

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[5] Equivalently, if we denote $x = c\rho^{-1}$, where c is a positive constant, $f_x(x) \doteq x^b$ means $\lim_{x\to 0} \frac{\log f_x(x)}{\log x} = b$, and $f(x) \dot{\leq} x^b$ means $\lim_{x\to 0} \frac{\log f_x(x)}{\log x} \geq b$.

Proof: see Appendix A.

Furthermore, the following results are helpful for analysis in Section III, supplying a diversity upper bound among all possible ordering rules.

Lemma II: *The maximal possible diversity order for the first decoded layer in V-BLAST systems is given by*

$$d_{1\_max}^{(VBLST)}(r) = \lim_{\rho_0 \to \infty} \frac{\log \Pr\left\{ \max_k R_{k,span\{\bar{k}\}} \le N_T \rho_0^{-\left(1-\frac{r}{N_T}\right)} \right\}}{\log(1/\rho_0)}, \tag{9}$$

where $R_{k,Span\{\bar{k}\}}$ *is the squared projection height from* $\mathbf{h}_k$ *to the space spanned by all the other* $N_T - 1$ *column vectors. It can be achieved by the ordering rule in* [1], *which results in the maximal post processing SNR at each detection step of the SIC receivers.*

The proof of Lemma II is straightforward, and (9) is obtained through (7) and (8). It is noteworthy that Lemma II serves to support the proof of Theorem I in Section III, which reveals the fact that this "maximal possible" diversity order (9) is actually not improved from the non-ordered case. On the other hand, the log-scale diversity expressions in (6) and (9) cannot measure the coding gains of MIMO systems, so layer ordering is still meaningful.

We further have that $R_{k,span\{\bar{k}\}}$ is independent of any vector in $span\{\bar{k}\}$, as a corollary of the following lemma.

Lemma III: *Let* $R_{k,span\{U_{\bar{k}}\}}$ *be the squared project height from a complex Gaussian i.i.d.* $N_R \times 1$ *random vector* $\mathbf{h}_k$ *, to any space* $span\{U_{\bar{k}}\}$ *, spanned by a set of* $1 \le L \le N_R - 1$ *random vectors* $\mathbf{h}_{j_1}, ..., \mathbf{h}_{j_L}$ *, all independent with* $\mathbf{h}_k$ *, then* $R_{k,span\{U_{\bar{k}}\}}$ *is independent of any vector in* $span\{U_{\bar{k}}\}$.

Proof: See Appendix B.

B. Multiple Access MIMO Channels

For a SDMA channel, multiple receive antennas can be used to separate signals from different users transmitted in the same frequency-time slot, leading to multiple access gains. As in [7], we assume that there are K co-channel transmitters each equipped with N_T^k antennas, and $N_T^{All} = \sum_{k=1}^{K} N_T^k \le N_R$ (this is the dimensionality requirement for ZF or MMSE SIC receivers). Consider a synchronous model[6] with equal power users for simplicity [7], (1) still applies, but $\mathbf{H}$ is now expressed as

$$\mathbf{H} = [\mathbf{H}^{(1)}, \mathbf{H}^{(2)}, ..., \mathbf{H}^{(K)}], \quad (10)$$

where $\mathbf{H}^{(k)} = [\mathbf{h}_1^{(k)}, \mathbf{h}_2^{(k)}, ..., \mathbf{h}_{N_T^k}^{(k)}]$ is an $N_R \times N_T^k$ sub-matrix corresponding to the channel between the kth transmitter and the receiver. For convenience in the following analysis, we also define

$$\mathbf{H}^{(\bar{k})} = [\mathbf{H}^{(1)}, .., \mathbf{H}^{(k-1)}, \mathbf{H}^{(k+1)}, ..., \mathbf{H}^{(K)}]. \quad (11)$$

The “^” notation introduced in II.A for normalized vectors also applies here. Also $\mathbf{s}$ becomes

$$\mathbf{s} = [\mathbf{s}_1^T,\ \mathbf{s}_2^T, \cdots, \mathbf{s}_K^T]^T, \quad (12)$$

where the $N_T^k \times t$ sub-matrix $\mathbf{s}_k$ is the data block from transmitter k. Data streams of one transmitter are allowed to be jointly encoded and decoded, while those from different transmitters are independent (which can be viewed as a *generalized layered structure*, each *general layer* containing more than one data streams). If $N_T^k = 1,\ \forall k$, SDMA MIMO coincides with point-to-point V-BLAST.

For simplicity, we will only consider the ZF-SIC receiver. Specifically, with fixed detection order (e.g., from user 1 to K), when detecting the data streams for the kth transmitter, we null out the interference from transmitters k+1 to K by projecting each column vector in $\mathbf{H}^{(k)}$ to the null space of the space spanned by the column vectors in $\left\{\mathbf{H}^{(k+1)}, ..., \mathbf{H}^{(K)}\right\}$, after cancelling the interference

[6] In practice, this may be achieved by timing-advance technologies, employed currently in uplink of GSM and 3G cellular networks [20].

contributed by transmitters 1 to k-1. In this scenario, the diversity-multiplexing tradeoff, representing the asymptotic exponential behavior of the probability that any user is erroneously detected, is given by [7]

$$d(r_1, r_2, ..., r_K) = \min_k d^*_{N_T^k,\, N_R - \sum_{j>k} N_T^j}(r_k), \qquad (13)$$

where $d^*_{m,n}(r) = (m-r)(n-r)$ stands for the optimal diversity-multiplexing tradeoff for an $n \times m$ Rayleigh fading MIMO channel [6], and r_k is the spatial multiplexing gain for user k. For ease of analysis, we focus on the symmetric scenario suggested by [7]: $r_k = r,\ N_T^k = N_T,\ \forall k$; extension to the non-symmetric scenario constitutes our future research. With these settings, (13) becomes

$$d^{(SDMA)}_{non-order}(r) = (N_T - r)(N_R - (K-1)N_T - r)^{7}. \qquad (14)$$

Suppose user k is chosen in the *first* decoding stage, its equivalent channel after the nulling operation can be expressed as:

$$\mathbf{R}^{(k)} = [\mathbf{r}^{(k)}_{1,span\{\mathbf{H}^{(\bar{k})}\}}, \mathbf{r}^{(k)}_{2,span\{\mathbf{H}^{(\bar{k})}\}}, ..., \mathbf{r}^{(k)}_{N_T,span\{\mathbf{H}^{(\bar{k})}\}}], \qquad (15)$$

where $\mathbf{r}^{(k)}_{j,span\{\mathbf{H}^{(\bar{k})}\}}$ is the projection from $\mathbf{h}^{(k)}_j$ to the null space of $\mathbf{H}^{(\bar{k})}$ (c.f. (11)). As indicated in [7], $\mathbf{R}^{(k)}$ is statistically equivalent to a point-to-point MIMO channel with N_T transmit and $N_R - (K-1)N_T$ receive antennas. We have the following analogue of Lemma II, whose proof is straightforward and omitted.

Lemma IV*: The maximal possible diversity order for the first decoded general layer in SDMA systems is given by*

$$d^{(SDMA)}_{1\_max}(r) = \lim_{\rho_0 \to \infty} \frac{\log \Pr\{\max_k C_k \le r \log \rho_0\}}{\log(1/\rho_0)}, \qquad (16)$$

[7] Note that in this multiple access context, we assume that data streams of one transmitter are jointly encoded and optimally decoded, while those from different transmitters are independent. When $K=1$, (14) reduces to the optimal diversity-multiplexing tradeoff of the single user channel, rather than that of single user V-BLAST in (17).

where $C_k = C(\mathbf{R}^{(k)}, \rho_0)$ *is the MIMO channel capacity with channel* $\mathbf{R}^{(k)}$ *and SNR level* ρ_0 [8]*. It can be achieved by the ordering rule that results in the maximal user capacity at each detection stage (or general layer).*

Similar to Lemma II, Lemma IV serves to support the proof of Theorem II in Section IV, which states that this "maximal possible" diversity order is actually the same as the non-ordered case with the fixed data rate, and this ordering rule is beneficial not for diversity improvement, but for coding gain.

It was claimed in [7] that the diversity analysis, when ordering is involved, is in general not accessible. We are able to solve it in Section IV, by applying the technique proposed in the following section.

III. Ordered V-BLAST SIC Receiver

In this section, adopting a novel geometrical approach, we rigorously prove the following Theorem:

Theorem I*: For general point-to-point V-BLAST systems with SIC receivers, the diversity-multiplexing tradeoff of the system is not improved by the optimal ordering.*

As indicated before, Theorem I is verified in [4][5][8] only for the simplest $N_T = 2$ case, while we will investigate the general scenario. Here the optimal ordering is defined from the error probability viewpoint. Clearly, an exhaustive search among all the $N_T!$ possible permutations targeting the minimum joint error probability leads to the optimal ordering rule, which is denoted as π, together with the corresponding joint error probability $P_{e\_\pi}(\rho_0)$ and diversity order $d_{opt}^{(VBLAST)}(r)$. In the following, we determine $d_{opt}^{(VBLAST)}(r)$ through tight bounding techniques.

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[8] Throughout this paper, we assume equal power allocation without loss of generality, as optimal power allocation doesn't increase the asymptotic exponential behavior of the outage probability [6].

First, the performance of a fixed-order V-BLAST SIC receiver [18][6] provides a good lower bound to $d_{opt}^{(VBLAST)}(r)$, i.e.,

$$d_{opt}^{(VBLAST)}(r) \geq d_{non-order}^{(VBLAST)}(r) = (N_R - N_T + 1)(1 - r / N_T)\,. \tag{17}$$

Secondly, let the error probability of the first decoded layer with the ordering rule π be $P_{1\_\pi}(\rho_0)$. By Lemma II, $P_{e\_\pi}(\rho_0) \geq P_{1\_\pi}(\rho_0) \geq P_{1\_maxsnr}(\rho_0)$, where $P_{1\_maxsnr}(\rho_0)$ stands for the error probability of the first decoded layer assuming the ordering rule in [1]. Therefore,

$$d_{opt}^{(VBLAST)}(r) \leq d_{1\_max}^{(VBLST)}(r)\,, \tag{18}$$

and we are left to evaluate (9).

The difficulty of such evaluation lies in that different $R_{k,Span\{\bar{k}\}}$ in (9) are inter-dependent, and the exact distribution of the post-processing SNR is in general not accessible. An exception is for the $N_T = 2$ case [4][5][8], for which one has (see Figure 1)

$$R_{1,2} = \| \mathbf{h}_1 \|^2 \sin^2 \theta_{12}\,, \text{ and } R_{2,1} = \| \mathbf{h}_2 \|^2 \sin^2 \theta_{12}\,,$$

where $R_{i,j} = R_{i,span\{j\}}$ is the simplified notation of the squared projection height from $\mathbf{h}_i$ to $\mathbf{h}_j$; and θ_{ij} is the angle between the two vectors $\mathbf{h}_i$ and $\mathbf{h}_j$, defined as $\theta_{ij} = \sin^{-1} \frac{\sqrt{R_{i,j}}}{\left\| \mathbf{h}_j \right\|}, 0 < \theta_{ij} < \frac{\pi}{2}$. The ordering rule of [1] is reduced to simply choosing the transmit antenna whose corresponding column vector has a larger norm, therefore the asymptotic exponential behavior of its outage probability can be explicitly explored. It is shown that in this case the diversity order is the same as that of the fixed-order case. However, this approach can not be extended to the $N_T > 2$ scenarios.

In the following we complete the proof of Theorem I by deriving a tight diversity upper bound for (9) that matches (17).

Lemma V: *With the above settings, we have*

$$d_{1\_\max}^{(VBLST)}(r) \le (N_R - N_T + 1)(1 - r/N_T)\,. \tag{19}$$

Proof: By (9) we mainly investigate the diversity order of the outage probability $\Pr\left\{\max_k R_{k,span\{\bar{k}\}} \le x\right\}$, with $x = N_T \rho_0^{-(1-r/N_T)}$.

Step 1:

At first, we decompose $\mathbf{h}_1$ by Gram-Schmidt orthogonalization [12] as (recall the "^" notation introduced in Section II):

$$\begin{aligned}\mathbf{h}_1 &= [\hat{\mathbf{h}}_{N_T},\quad \hat{\mathbf{h}}_{N_T-1,span\{N_T\}},\quad \ldots,\quad \hat{\mathbf{h}}_{2,span\{3,..,N_T\}}]\boldsymbol{\alpha} + \sqrt{R_{1,span\{2,\ldots,N_T\}}}\,\hat{\mathbf{h}}_{1,span\{2,\ldots,N_T\}} \\ &= \hat{\mathbf{H}}_{\bar{1}\_orth}\boldsymbol{\alpha} + \sqrt{R_{1,span\{2,\ldots,N_T\}}}\,\hat{\mathbf{h}}_{1,span\{2,\ldots,N_T\}} \\ &= \hat{\mathbf{H}}_{\bar{1}}\boldsymbol{\gamma} + \sqrt{R_{1,span\{\bar{1}\}}}\,\hat{\mathbf{h}}_{1,span\{\bar{1}\}},\end{aligned} \tag{20}$$

where $\hat{\mathbf{h}}_{k,\Omega}$ is the unit vector along the projection of $\mathbf{h}_k$ on the null space of the vector subspace Ω; the vector $\boldsymbol{\alpha}$ contains the set of coordinates when $\mathbf{h}_1$ is projected on the orthonormal basis $\hat{\mathbf{H}}_{\bar{1}\_orth} = [\hat{\mathbf{h}}_{N_T},\quad \ldots\quad \hat{\mathbf{h}}_{2,span\{3,..,N_T\}}]$ for the subspace $span\{2,3,\ldots N_T\} \triangleq span\{\bar{1}\}$. The last line of (20) denotes coordinate transformation from $\hat{\mathbf{H}}_{\bar{1}\_orth}$ to $\hat{\mathbf{H}}_{\bar{1}} = [\hat{\mathbf{h}}_{N_T},\quad \hat{\mathbf{h}}_{N_T-1},\quad \ldots\quad \hat{\mathbf{h}}_2]$ with new coordinates given by

$$\boldsymbol{\gamma} = [\gamma_{N_T},\ldots,\gamma_2]^T = \mathbf{Q}\boldsymbol{\alpha}\,, \tag{21}$$

where $\mathbf{Q}$ is the transfer matrix [12], independent with $R_{1,span\{\bar{1}\}}$ by Lemma III. As $\boldsymbol{\alpha}$ is also independent with $R_{1,span\{\bar{1}\}}$ (coordinates of a Gaussian random vector on an orthonormal basis are jointly independent), so is $\boldsymbol{\gamma}$.

Figure 2 illustrates the $N_T = 3$ scenario. Here the decomposition of $\mathbf{h}_1$ is

$$\begin{aligned}\mathbf{h}_1 &= \begin{bmatrix}\hat{\mathbf{h}}_3 & \hat{\mathbf{h}}_{2,span\{3\}}\end{bmatrix}\boldsymbol{\alpha} + \sqrt{R_{1,span\{2,3\}}}\hat{\mathbf{h}}_{1,span\{2,3\}} \\ &= \begin{bmatrix}\hat{\mathbf{h}}_3 & \hat{\mathbf{h}}_2\end{bmatrix}\boldsymbol{\gamma} + \sqrt{R_{1,span\{2,3\}}}\hat{\mathbf{h}}_{1,span\{2,3\}},\end{aligned} \tag{22}$$

where according to the *change of vector coordinate law* in [12],

$$\boldsymbol{\gamma} = \begin{bmatrix} 1 & -\dfrac{\cos\theta_{23}}{\sin\theta_{23}} \\ 0 & \dfrac{1}{\sin\theta_{23}} \end{bmatrix}\boldsymbol{\alpha}. \tag{23}$$

Step 2:

Secondly, we intend to build up the relationship of $R_{2,span\{\bar{2}\}}, R_{3,span\{\bar{3}\}}, ..., R_{N_T,span\{\bar{N}_T\}}$ with $R_{1,span\{\bar{1}\}}$. Since $\sqrt{R_{2,span\{\bar{2}\}}}$ is the magnitude of the projection from $\mathbf{h}_2$ to the null space of $span\{\bar{2}\} \triangleq span\{1,3,4,..,N_T\}$, it represents the shortest distance from vector $\mathbf{h}_2$ to any points in the subspace $span\{\bar{2}\}$. That is

$$\begin{aligned}R_{2,span\{\bar{2}\}} &= \min_{\boldsymbol{\beta}}\left\|\mathbf{h}_2 - [\mathbf{h}_1, \hat{\mathbf{h}}_3, ..., \hat{\mathbf{h}}_{N_T}]\boldsymbol{\beta}\right\|^2 = \min_{\boldsymbol{\beta}}\left\|\sqrt{R_2}\hat{\mathbf{h}}_2 - [\mathbf{h}_1, \hat{\mathbf{h}}_3, ..., \hat{\mathbf{h}}_{N_T}]\boldsymbol{\beta}\right\|^2 \\ &= \min_{\boldsymbol{\beta}}\left\|\sqrt{R_2}\hat{\mathbf{h}}_2 - \beta_1\left(\sum_{j=2}^{N_T}\gamma_j\hat{\mathbf{h}}_j + \sqrt{R_{1,span\{\bar{1}\}}}\hat{\mathbf{h}}_{1,span\{\bar{1}\}}\right) - \sum_{i=3}^{N_T}\beta_i\hat{\mathbf{h}}_i\right\|^2,\end{aligned} \tag{24}$$

where $\boldsymbol{\beta} = [\beta_1, \beta_3, .., \beta_{N_T}]^T$, such that $[\mathbf{h}_1, \hat{\mathbf{h}}_3, ..., \hat{\mathbf{h}}_{N_T}]\boldsymbol{\beta}$ represents an arbitrary vector in $span\{\bar{2}\}$; $R_k \triangleq \|\mathbf{h}_k\|^2$, $\forall k$; and for the third equality we replace $\mathbf{h}_1$ by (20). Setting $\beta_1 = \sqrt{R_2}/\gamma_2$ (c.f. (21)) and $\beta_k = -\gamma_k\beta_1$, $3 \le k \le N_T$, in the right hand side of (24), we have

$$R_{2,span\{\bar{2}\}} \le \left\|\beta_1\sqrt{R_{1,span\{\bar{1}\}}}\hat{\mathbf{h}}_{1,span\{\bar{1}\}}\right\|^2 = \beta_1^2 R_{1,span\{\bar{1}\}} = \frac{R_2}{\gamma_2^2}R_{1,span\{\bar{1}\}}. \tag{25}$$

In another word, we upper bound $R_{2,span\{\bar{2}\}}$ by (non-orthogonally) projecting $\mathbf{h}_2$ onto $span\{\bar{2}\}$ along the direction of $\hat{\mathbf{h}}_{1,span\{\bar{1}\}}$. Following a similar approach, we have

$$R_{k,span\{\bar{k}\}} \le \frac{R_k}{\gamma_k^2} R_{1,span\{\bar{1}\}}, \; 2 \le k \le N_T. \tag{26}$$

Step 3:

Finally through (26):

$$\begin{aligned}
\Pr\left\{\max_k R_{k,span\{\bar{k}\}} \le x\right\} &= \Pr\left\{R_{1,span\{\bar{1}\}} \le x, R_{2,span\{\bar{2}\}} \le x, \ldots, R_{N_T,span\{\bar{N}_T\}} \le x\right\} \\
&\ge \Pr\left\{R_{1,span\{\bar{1}\}} \le x, \frac{R_2}{\gamma_2^2} R_{1,span\{\bar{1}\}} \le x, \ldots, \frac{R_{N_T}}{\gamma_{N_T}^2} R_{1,span\{\bar{1}\}} \le x\right\} \\
&\ge \Pr\left\{R_{1,span\{\bar{1}\}} \le x, \frac{R_2}{\gamma_2^2} \le 1, \ldots, \frac{R_{N_T}}{\gamma_{N_T}^2} \le 1\right\} \\
&= \Pr\left\{R_{1,span\{\bar{1}\}} \le x\right\} \Pr\left\{\frac{R_2}{\gamma_2^2} \le 1, \ldots, \frac{R_{N_T}}{\gamma_{N_T}^2} \le 1\right\},
\end{aligned} \tag{27}$$

where the last equality comes from the independence between $\left\{R_2/\gamma_2^2, \ldots, R_{N_T}/\gamma_{N_T}^2\right\}$ and $R_{1,span\{\bar{1}\}}$ (c.f. Lemma III and Step 1). The second term in the last line of (27) will only introduce a non-vanishing constant factor, as it is not a function of x, therefore

$$\Pr\left\{\max_k R_{k,span\{\bar{k}\}} \le x\right\} \dot{\ge} \Pr\left\{R_{1,span\{\bar{1}\}} \le x\right\} \doteq \rho_0^{-d_{non-order}^{(VBLAST)}(r)}.$$

This concludes the proof of Lemma V.

□

IV. Ordered SDMA SIC Receiver

As introduced in Section II-B, in SDMA channels multiple data streams for one user are allowed to be jointly encoded in space and time. The diversity analysis with ordering is more involved in this case than in point-to-point channels, since different capacities are correlated in a more complicated manner. This section presents our method for getting around this problem, and gives the proof of the following Theorem:

Theorem II: *For symmetric SDMA systems with ZF-SIC receivers as defined in Section II-B, the following lower and upper bounds of the diversity-multiplexing tradeoff hold for the optimal ordering rule:*

$$d_L(r) \le d(r) \le d_U(r), \tag{28}$$

where $d_L(r) = (N_T - r)(N_R - (K-1)N_T - r)$, *and* $d_U(r) = N_T(N_R - (K-1)N_T)(1 - r/N_T)$.

Proof: Here we follow similar steps as the proof of Theorem I: at first, since the optimally ordered SDMA SIC receiver will always outperform the non-ordered case, we have

$$d_{opt}^{(SDMA)}(r) \ge d_{non-order}^{(SDMA)}(r) = d_L(r). \tag{29}$$

Secondly, since joint error probability of the optimally ordered SDMA receiver is no smaller than that of its first decoded user (general layer), together with Lemma IV, we have

$$d_{opt}^{(SDMA)}(r) \le d_{1\_max}^{(SDMA)}(r). \tag{30}$$

In the following, we verify that the diversity order of (16) is upper bounded by $d_U(r)$. It is easily seen that

$$C_k = C(\mathbf{R}^{(k)}, \rho_0) = \sum_{n=1}^{N_T} \log\left(1 + \frac{\rho_0}{N_T}\lambda_n^{(k)}\right) \le N_T \log\left(1 + \frac{\rho_0}{N_T}\left\|\mathbf{R}^{(k)}\right\|_F^2\right), \tag{31}$$

where $\left\{\lambda_n^{(k)}\right\}$ are eigenvalues of the Hermitian matrix $\left(\mathbf{R}^{(k)}\right)^H \mathbf{R}^{(k)}$, and $\| \, \|_F$ represents the Frobenius norm. Then

$$\Pr\left\{\max_k C_k \le r\log\rho_0\right\} \dot{\ge} \Pr\left\{\max_k \left\|\mathbf{R}^{(k)}\right\|_F^2 \le \rho_0^{(r/N_T - 1)}\right\}. \tag{32}$$

The right hand side of (32) can be rewritten as

$$\Pr\left(\max_{k=1\ldots K}\left\|\mathbf{R}^{(k)}\right\|_F^2 \le x\right) = \Pr\left(\sum_{j=1}^{N_T} R_{j,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)} \le x, \sum_{j=1}^{N_T} R_{j,span\{\mathbf{H}^{(\bar{2})}\}}^{(2)} \le x, \ldots, \sum_{j=1}^{N_T} R_{j,span\{\mathbf{H}^{(\bar{K})}\}}^{(K)} \le x\right), \tag{33}$$

where $R_{j,span\{\mathbf{H}^{(\bar{K})}\}}^{(k)}$ stands for the squared norm of $\mathbf{r}_{j,span\{\mathbf{H}^{(\bar{k})}\}}^{(k)}$ (c.f. (15)), and $x = \rho_0^{(r/N_T - 1)}$.

Similar to (20), first we decompose $\mathbf{h}_j^{(1)}$ by (recall the "^" notation introduced in Section II):

$$\mathbf{h}_j^{(1)} = \hat{\mathbf{H}}^{(\bar{1})}\boldsymbol{\gamma}_j^{(1)} + \sqrt{R_{j,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)}}\hat{\mathbf{h}}_{j,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)}, \forall j = 1...N_T, \quad (34)$$

where $\hat{\mathbf{h}}_{j,span\{\mathbf{H}^{(\bar{k})}\}}^{(k)}$ is the unit vector along the projection of $\mathbf{h}_j^{(k)}$ on the null space of $span\{\mathbf{H}^{(\bar{k})}\}$; and $\boldsymbol{\gamma}_j^{(k)}$ collects the coordinates for the projection of $\mathbf{h}_j^{(k)}$ on $\hat{\mathbf{H}}^{(\bar{k})}$, which is independent with $R_{j,span\{\mathbf{H}^{(\bar{k})}\}}^{(k)}$.

Secondly, the relationships of $\{R_{j,span\{\mathbf{H}^{(\bar{2})}\}}^{(2)}\}_{\forall j=1..N_T}, \{R_{j,span\{\mathbf{H}^{(\bar{3})}\}}^{(3)}\}_{\forall j=1..N_T}, ..., \{R_{j,span\{\mathbf{H}^{(\bar{K})}\}}^{(K)}\}_{\forall j=1..N_T}$ with $\{R_{j,span\{\mathbf{H}^{(\bar{2})}\}}^{(1)}\}_{\forall j=1..N_T}$ can be built up through a similar method as in (24):

$$\begin{aligned} R_{j,span\{\mathbf{H}^{(\bar{k})}\}}^{(k)} &= \min_{\boldsymbol{\beta}_j^{(k)}} \left\| \sqrt{R_j^{(k)}}\hat{\mathbf{h}}_j^{(k)} - [\mathbf{H}^{(1)}, \hat{\mathbf{H}}^{(2)}, ..., \hat{\mathbf{H}}^{(k-1)}, \hat{\mathbf{H}}^{(k+1)}, ..., \hat{\mathbf{H}}^{(K)}]\boldsymbol{\beta}_j^{(k)} \right\|^2 \\ &= \min_{\boldsymbol{\beta}_j^{(k)}} \left\| \sqrt{R_j^{(k)}}\hat{\mathbf{h}}_j^{(k)} - \sum_{i=1}^{N_T} \beta_{j\_1i}^{(k)} \left(\hat{\mathbf{H}}^{(\bar{1})}\boldsymbol{\gamma}_i^{(1)} + \sqrt{R_{i,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)}}\hat{\mathbf{h}}_{i,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)} \right) - \sum_{\substack{l=2 \\ l\neq k}}^{K} \sum_{i=1}^{N_T} \beta_{j\_li}^{(k)}\hat{\mathbf{h}}_i^{(l)} \right\|^2 \\ &\leq \left\| \sum_{i=1}^{N_T} \overline{\beta}_{j\_1i}^{(k)} \sqrt{R_{i,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)}}\hat{\mathbf{h}}_{i,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)} \right\|^2 \leq \sum_{i=1}^{N_T} \overline{\beta}_{j\_1i}^{(k)\,2} R_{i,span\{\mathbf{H}^{(\bar{1})}\}}^{(1)}, \quad \forall k > 1,\ j = 1...N_T \end{aligned} \quad (35)$$

where $[\mathbf{H}^{(1)}, \hat{\mathbf{H}}^{(2)}, ..., \hat{\mathbf{H}}^{(k-1)}, \hat{\mathbf{H}}^{(k+1)}, ..., \hat{\mathbf{H}}^{(K)}]\boldsymbol{\beta}_j^{(k)}$ represents an arbitrary vector in the space $span\{\mathbf{H}^{(\bar{k})}\}$; $\boldsymbol{\beta}_j^{(k)} = [\beta_{j\_11}^{(k)}, ..., \beta_{j\_1N_T}^{(k)}, \beta_{j\_21}^{(k)}, ..., \beta_{j\_(k-1)N_T}^{(k)}, \beta_{j\_(k+1)1}^{(k)}, ..., \beta_{j\_KN_T}^{(k)}]^T$ collects the coefficients of such a vector on the basis $[\mathbf{H}^{(1)}, \hat{\mathbf{H}}^{(2)}, ..., \hat{\mathbf{H}}^{(k-1)}, \hat{\mathbf{H}}^{(k+1)}, ..., \hat{\mathbf{H}}^{(K)}]$ of $span\{\mathbf{H}^{(\bar{k})}\}$, where $\beta_{j\_li}^{(k)}$ stands for the coefficient on the *i-th* column vector in $\hat{\mathbf{H}}^{(l)}$ (or $\mathbf{H}^{(l)}$ when $l = 1$) ; and $R_j^{(k)} = \left\|\mathbf{h}_j^{(k)}\right\|^2$. For the second equality we plug in (34) for columns of $\mathbf{H}^{(1)}$. The first inequality in (35) is obtained by assigning the coefficients in $\boldsymbol{\beta}_j^{(k)}$ particular values: $\boldsymbol{\beta}_j^{(k)} = \overline{\boldsymbol{\beta}}_j^{(k)}$, such that the coefficients of all the vectors $\{\hat{\mathbf{h}}_i^{(l)}\}_{l=2...K, i=1...N_T}$ in (35) are zero, and the last inequality is due to the triangle inequality. As what we argue in Section III, any element in $\overline{\boldsymbol{\beta}}_j^{(k)}$ is determined only by $\{R_j^{(k)}\}_{k=2...N_T}$ and $\{\boldsymbol{\gamma}_i^{(1)}\}_{i=1...N_T}$. Hence $\{\overline{\beta}_{j\_1i}^{(k)}\}_{i=1...N_T}$ are independent with $\left\{R^{(1)}_{i,span\{\mathbf{H}^{(\bar{1})}\}}\right\}_{i=1\ldots N_T}$, since both $\{R^{(k)}_j\}_{k=2\ldots N_T}$ and $\{\boldsymbol{\gamma}^{(1)}_i\}_{i=1\ldots N_T}$ are determined by vectors inside $span\{\mathbf{H}^{(\bar{1})}\}$ (c.f. Lemma III). For example, in the case of $K=2, N_T=2$, we get

$$\begin{bmatrix} \bar{\beta}^{(2)}_{j\_11} \\ \bar{\beta}^{(k)}_{j\_12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\gamma}^{(1)}_1 & \boldsymbol{\gamma}^{(1)}_2 \end{bmatrix}^{-1} \begin{bmatrix} R^{(2)}_j \\ 0 \end{bmatrix}, \; j=1,2. \tag{36}$$

Now given (35), we are able to lower bound (33) by a similar approach as in (27):

$$\begin{aligned} &\Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x, \sum_{j=1}^{N_T} R^{(2)}_{j,span\{\mathbf{H}^{(\bar{2})}\}} \le x, \ldots, \sum_{j=1}^{N_T} R^{(K)}_{j,span\{\mathbf{H}^{(\bar{K})}\}} \le x\right) \\ &\ge \Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x, \sum_{j=1}^{N_T}\left(\sum_{i=1}^{N_T} \bar{\beta}^{(2)\,2}_{i\_1j}\right) R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x, \ldots, \sum_{j=1}^{N_T}\left(\sum_{i=1}^{N_T} \bar{\beta}^{(K)2}_{i\_1j}\right) R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x\right) \quad (37) \\ &\ge \Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x, \sum_{j=1}^{N_T}\sum_{i=1}^{N_T} \bar{\beta}^{(2)\,2}_{i\_1j} \le 1, \ldots, \sum_{j=1}^{N_T}\sum_{i=1}^{N_T} \bar{\beta}^{(K)2}_{i\_1j} \le 1\right) \\ &= \Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x\right) \Pr\left(\sum_{j=1}^{N_T}\sum_{i=1}^{N_T} \bar{\beta}^{(2)\,2}_{i\_1j} \le 1, \ldots, \sum_{j=1}^{N_T}\sum_{i=1}^{N_T} \bar{\beta}^{(K)2}_{i\_1j} \le 1\right) \\ &\doteq \Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le x\right), \end{aligned}$$

where we explore the independence between $\{\bar{\beta}^{(k)}_{j\_1i}\}_{i,j=1\ldots N_T,\, k>1}$ and $\left\{R^{(1)}_{i,span\{\mathbf{H}^{(\bar{1})}\}}\right\}_{i=1\ldots N_T}$ as illustrated above.

As $\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}}$ is of χ^2 distribution with the degree of freedom of $2N_T(N_R-(K-1)N_T)$, we have (c.f. (32))

$$\Pr\left\{\max_k C_k \le r\log\rho_0\right\} \dot{\ge} \Pr\left(\sum_{j=1}^{N_T} R^{(1)}_{j,span\{\mathbf{H}^{(\bar{1})}\}} \le \rho_0^{(r/N_T-1)}\right) \doteq \rho_0^{-N_T[N_R-(K-1)N_T](1-r/N_T)} = \rho_0^{-d_U(r)} \; . \tag{38}$$

This concludes the proof of Theorem II.

□

Remarks: Figure 3 shows the upper and lower bounds of the tradeoff curve given by Theorem II. Note that the point "$d(0)$ upper bound in [7]", given by $N_T(N_R-N_T)$, is derived by a genie-aided method in [7]. Therefore our result provides a tighter upper bound. More importantly, in the scenario with *fixed data rates* ($r = 0$), Theorem II tells us that the optimal ordering will not increase the diversity order of the joint error probability for the K users when SIC receivers are deployed. In another word, *the diversity order with fixed data rate is unchanged by the optimum user ordering*.

Given that the two extreme points, corresponding to $r = 0$ and $d(r) = 0$, respectively, are unchanged from non-ordering case, we conjecture that the whole diversity-multiplexing tradeoff curve will not be improved by optimal ordering, although the accurate analysis for the intermediate points with $r > 0$ is still challenging.

V. Conclusions

In this paper, we propose a novel geometry-based method to analyze the diversity-multiplexing tradeoff for V-BLAST SIC receivers deployed in point-to-point and SDMA systems. Our results rigorously show that for point-to-point channels, the tradeoff curve is not changed by ordering; while for SDMA channels, we derive a tighter upper bound of the tradeoff curve than that in [7], and we prove that the diversity order with fixed data rate is not improved by user ordering.

Finally we stress again that the diversity-multiplexing tradeoff is sometimes a loose indication of the system performance, as it only characterizes the exponential behavior of the error probability. Although optimal ordering does not improve the diversity order of a V-BLAST type system with SIC receivers, it still can provide a coding gain as compared to fixed-order SIC receivers.

Appendix

A. Proof of Lemma I

First we show that the diversity order is determined by the decoded layer with the maximum error probability (assuming perfect feedback), for either ZF- or MMSE- SIC receivers, and for any ordering rule. Assuming no interference from the previously decoded layers, we define $P_e^{(l)}$ the average error probability of the lth decoded layer and P_e the system joint error probability as before, and define $P_e^{(1\sim l)}$ as the joint error probability of layers 1 to l. Clearly, $P_e \geq \max\left(P_e^{(1)}, P_e^{(2)}, ..., P_e^{(N_T)}\right)$, and

$$P_e^{(1\sim l)} = P_e^{(l)}(1-P_e^{(1\sim l-1)}) + P_e^{(1\sim l-1)}. \tag{39}$$

For the second decoded layer, (39) becomes

$$P_e^{(1\sim 2)} = P_e^{(2)}(1-P_e^{(1)}) + P_e^{(1)} \leq P_e^{(1)} + P_e^{(2)}. \tag{40}$$

By iteratively applying the above analysis in (39), we finally get $P_e = P_e^{(1\sim N_T)} \leq \sum_{l=1}^{N_T} P_e^{(l)}$. We then have $P_e \doteq \max\left(P_e^{(1)}, P_e^{(2)}, ..., P_e^{(N_T)}\right)$.

Therefore, to prove the Lemma, we are left to verify the diversity equivalence between ZF- and MMSE- SIC receivers in each layer assuming perfect feedback, i.e., $P_{e,ZF-SIC}^{(l)} \doteq P_{e,MMSE-SIC}^{(l)}, \forall l$. From discussion in Section II, it is then equivalent to show

$$\Pr(\rho_l^{(ZF-SIC)} \leq \rho_0^{r/N_T}) \doteq \Pr(\rho_l^{(MMSE-SIC)} \leq \rho_0^{r/N_T}),\ \forall l. \tag{41}$$

We assume multiplexing gain $r=0$ for simplicity, and the following analysis is readily extended to $0<r<N_T$ by replacing ρ_0 with $\rho_0^{(1-r/N_T)}$. In the trivial case $r=N_T$ we have zero diversity gain in both cases. Our following analysis is similar to the proof of Theorem IV.2 in [16], and we include it here mainly for completeness. In particular, we alternatively have expressions of post-processing SNRs as [14][18]

$$\rho_l^{(ZF-SIC)} = \frac{\rho_0}{(N_T-l+1)}\left(\mathbf{h}_{(l)}\right)^H\left[\mathbf{I}-\mathbf{H}_{(l)}\left(\left(\mathbf{H}_{(l)}\right)^H\mathbf{H}_{(l)}\right)^{-1}\left(\mathbf{H}_{(l)}\right)^H\right]\mathbf{h}_{(l)}, \tag{42}$$

$$\rho_l^{(MMSE-SIC)} = \frac{\rho_0}{(N_T-l+1)}\left(\mathbf{h}_{(l)}\right)^H\left[\mathbf{I}-\mathbf{H}_{(l)}\left(\left(\mathbf{H}_{(l)}\right)^H\mathbf{H}_{(l)}+\frac{(N_T-l+1)}{\rho_0}\mathbf{I}\right)^{-1}\left(\mathbf{H}_{(l)}\right)^H\right]\mathbf{h}_{(l)}, \tag{43}$$

where $\mathbf{H}_{(l)} = [\mathbf{h}_{(l+1)} \quad ... \quad \mathbf{h}_{(N_T)}]$. By matrix inversion Lemma, with $\mathbf{A}_{(l)} = \left(\left(\mathbf{H}_{(l)}\right)^H\mathbf{H}_{(l)}\right)^{-1}$, we have

$$\rho_l^{(MMSE-SIC)} - \rho_l^{(ZF-SIC)} = \left(\mathbf{h}_{(l)}\right)^H\mathbf{H}_{(l)}\mathbf{A}_{(l)}\left(\frac{(N_T-l+1)}{\rho_0}\mathbf{A}_{(l)}+\mathbf{I}\right)^{-1}\mathbf{A}_{(l)}\left(\mathbf{H}_{(l)}\right)^H\mathbf{h}_{(l)} \overset{\Delta}{=} \eta_{(l)}(\rho_0). \tag{44}$$

Clearly $\eta_{(l)}(\rho_0) \geq 0$ and thus

$$\Pr(\rho_l^{(MMSE-SIC)} \leq 1) \dot{\leq} \Pr(\rho_l^{(ZF-SIC)} \leq 1). \tag{45}$$

Another key observation is that, $\eta_{(l)}(\rho_0)$ is statistically independent with $\rho_l^{(ZF-SIC)}$, as the latter is proportional to the squared norm of the projection of $\mathbf{h}_{(l)}$ onto the null space of $\left(\mathbf{H}_{(l)}\right)^T$, while the former is the correlation of $\mathbf{h}_{(l)}$ with a vector in the range of $\mathbf{H}_{(l)}$ [13][15]. Furthermore, as $\rho_0 \to \infty$, it can be shown that

$$\eta_{(l)}(\rho_0) \overset{a.s.}{\to} \eta_{(l)}, \tag{46}$$

i.e., it converges almost surely to a positive random variable $\eta_{(l)}$ with finite mean. For our purpose, it is sufficient to know that $\Pr(\eta_{(l)} \leq 1/2)$ is a positive probability of $O(1)$ (the same order as 1). We thus have

$$\begin{aligned} \frac{\log \Pr(\rho_l^{(MMSE-SIC)} \leq 1)}{\log(1/\rho_0)} &= \frac{\log \Pr(\rho_l^{(ZF-SIC)} + \eta_{(l)}(\rho_0) \leq 1)}{\log(1/\rho_0)}, \\ &\geq \frac{\log \Pr(\rho_l^{(ZF-SIC)} \leq 1/2)}{\log(1/\rho_0)} + \frac{\log \Pr(\eta_{(l)}(\rho_0) \leq 1/2)}{\log(1/\rho_0)}, \end{aligned} \tag{47}$$

and

$$\lim_{\rho_0 \to \infty} \frac{\log \Pr(\eta_{(l)}(\rho_0) \leq 1/2)}{\log(1/\rho_0)} = \lim_{\rho_0 \to \infty} \frac{\log \Pr(\eta_{(l)} \leq 1/2)}{\log(1/\rho_0)} = 0, \tag{48}$$

where (48) is due to the fact that almost sure convergence always leads to convergence in distribution [19]. (47) and (48) together gives rise to

$$\Pr(\rho_l^{(MMSE-SIC)} \leq 1) \dot{\geq} \Pr(\rho_l^{(ZF-SIC)} \leq 1), \tag{49}$$

and (41) follows with (45) and (49).

B. Proof of Lemma III

From the definition, $R_{k,span\{U_{\bar{k}}\}} = \left\|\mathbf{P}\mathbf{h}_k\right\|^2$, where $\mathbf{P} = \mathbf{I} - \mathbf{B}\mathbf{B}^H$ is the projection matrix to the orthogonal space of $span\{U_{\bar{k}}\}$ [12], and $\mathbf{B}$ is composed of any orthonormal basis of this subspace. Since any projection matrix is *idempotent*, i.e. $\mathbf{P}^2 = \mathbf{P}$, its eigenvalues are either 1 or 0. Noting that $\mathbf{P}$ is also Hermitian, we can then write the eigenvalue decomposition of $\mathbf{P}$ as $\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, where $\mathbf{V}$ is unitary, and $\mathbf{\Lambda} = diag(1^{N_R - N_T + 1}, 0^{N_T - 1})$. Therefore

$$R_{k,span\{U_{\bar{k}}\}} = \left\| \mathbf{V}\mathbf{\Lambda}\mathrm{V}^H \mathbf{h}_k \right\|^2 = \left\| \mathbf{\Lambda}\mathbf{V}^H \mathbf{h}_k \right\|^2, \tag{50}$$

where the second equality follows due to the fact that a unitary transformation preserves length. From the definition of $\mathbf{P}$, as $\mathbf{h}_{j_1},...,\mathbf{h}_{j_L}$ are independent of $\mathbf{h}_k$, so is the unitary matrix $\mathbf{V}^H$. Therefore the conditional probability density function (PDF)

$$f\left(R_{k,span\{U_{\bar{k}}\}} \mid span\{U_{\bar{k}}\}\right) = f\left(\left\| \mathbf{\Lambda}\mathbf{V}_0^H \mathbf{h}_k \right\|^2\right) = f\left(\left\| \mathbf{\Lambda}\mathbf{h}_k \right\|^2\right) = f\left(R_{k,span\{U_{\bar{k}}\}}\right), \tag{51}$$

where $\mathbf{V}_0{}^H$ is a fixed unitary matrix dependent on the given realization of $span\{U_{\bar{k}}\}$, and the second equality comes from the rotationally invariant property of the i.i.d. complex Gaussian vector $\mathbf{h}_k$ [13][15]. Therefore, $R_{k,span\{U_{\bar{k}}\}}$ is independent of any vector in $span\{U_{\bar{k}}\}$.

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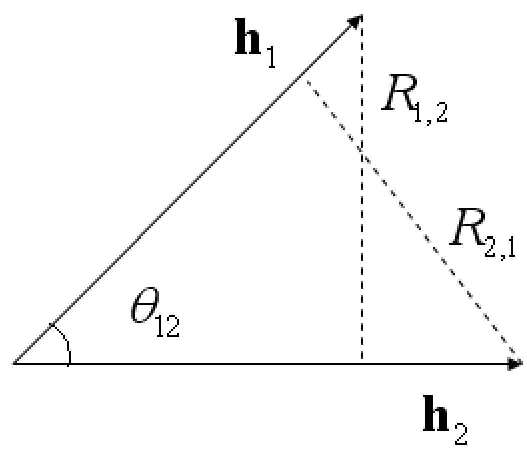

Figure 1 Geometric Illustration of ZF-SIC Receiver with $N_T = 2$

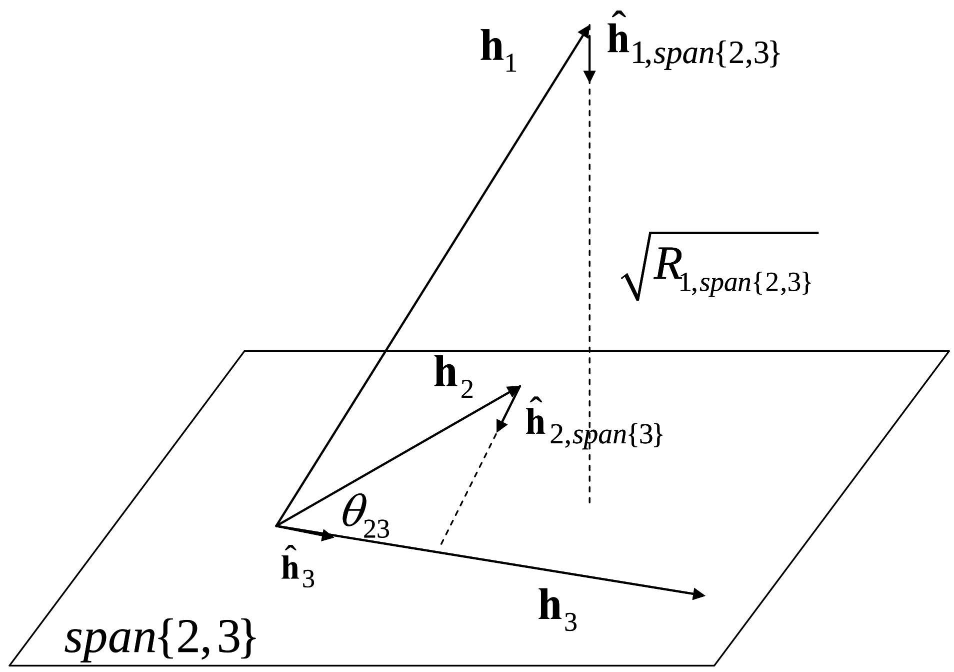

Figure 2 Geometric Illustration of the Decomposition of $\mathbf{h}_1$ with $N_T = 3$

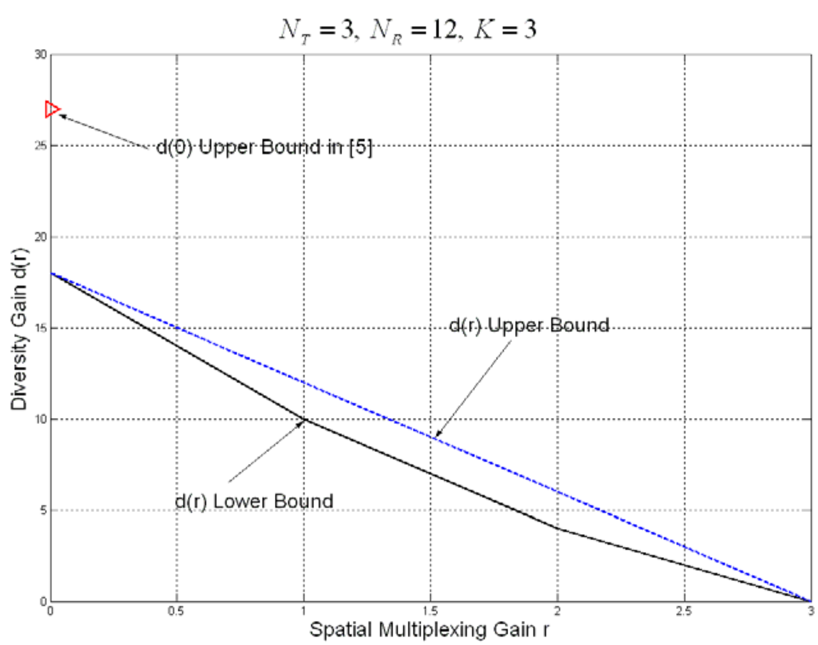

Figure 3 Bounds on the Diversity-Multiplexing Tradeoff for SDMA SIC Detectors