Asymptotic Analysis in MIMO MRT/MRC Systems

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Abstract

In this paper, through the analysis of the probability density function of the squared largest singular value of a complex Gaussian matrix at the origin and tail, we obtain two asymptotic results related to the multi-input multi-output (MIMO) maximum-ratio-transmission/maximum-ratio-combining (MRT/MRC) systems. One is the asymptotic error performance (in terms of SNR) in a single-user system, and the other is the asymptotic system capacity (in terms of the number of users) in the multiuser scenario when multiuser diversity is exploited. Similar results are also obtained for two other MIMO diversity schemes, space-time block coding and selection combining. Our results reveal a simple connection with system parameters, providing good insights for the design of MIMO diversity systems.

Index Terms: MIMO, MRT/MRC, Multiuser Diversity, Spatial Diversity

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I. Introduction

Multi-input multi-output (MIMO) systems can be exploited for spatial multiplexing or diversity gains. For a MIMO diversity system, appropriate diversity combining techniques are employed at the transmit and receive end to effectively transform the MIMO channel into an equivalent single-input single-output (SISO) one, with increased robustness. Depending on whether the channel state information (CSI) is required at the transmitter, MIMO diversity schemes can be divided into two categories: open-loop and closed-loop. Among the former is the scheme that employs well-known space-time block coding at the transmitter and maximum ratio combining at the receiver, coined as STBC/MRC. As certain feedback often exists in a wireless network (e.g., in use scheduling discussed below), closed-loop schemes are also of great interest. This category includes simple selection combining on both ends (SC/SC), joint maximum ratio transmission and maximum ratio combining (MRT/MRC), and various hybrid selection combining schemes in between.

For diversity usage, MRT/MRC systems provide the optimal performance reference [1]-[5], but its analysis is also more involved than others (see relevant distribution functions in Section II), which will be the focus of this paper. With the assumption that the receive beamforming vector is matched to the transmit one with unit modulus for all entries, the average output signal-to-noise ratio (SNR) of a MRT/MRC system is upper and lower bounded in [1], based on which the average symbol error rate (SER) and diversity order for a BPSK system are approximately derived. With the restricting assumptions in [1] removed, it is known that (for white Gaussian noise) the optimal transmit and receive beamformer are given by the principal right and left singular vector of the channel matrix $\mathbf{H}$, respectively; and the MIMO channel is transformed into a SISO link with equivalent channel gain $\sigma_{\max}$, the largest singular value of $\mathbf{H}$. For Rayleigh fading channels, the distribution of $\sigma_{\max}^2$, already derived in [6], is revisited in [2] and expressed in an alternative form – a linear combination of Gamma functions. Based on this expression, the exact system SER is derived for general modulation schemes in [2]. The distribution of $\sigma_{\max}^2$ for Ricean fading is obtained in [4]. Unfortunately, results in [2] and [4] don't easily lead one to an insightful understanding of the impact of the system parameters, including the number of transmit and receive antennas M and N, on performance. For example, in [2], the authors make two observations on MIMO MRT/MRC systems through simulation results: one is that when $M+N$ keeps fixed, the antennas distribution with $|M-N|$ minimized will provide the lowest SER, while the other is that when $M \times N$ is fixed, a distribution with the largest $M+N$ gives the best performance. But the authors do not provide a rigorous justification for both observations. Some similar observations are also made in [4].

In a multiuser wireless network, there is another form of diversity called multiuser diversity, which reflects the fact of independent fluctuations of different users' channels [7]. Multiuser diversity can be exploited to increase the system throughput, through intentionally transmitting to the user(s) with good channels at each instant (opportunistic scheduling). There exist some work on the joint spatial diversity and multiuser diversity systems. In particular, the system capacity analysis for Rayleigh fading channels is given in [8], and in [9] for more general Nakagami fading channels. While these results are accurate, simpler expressions are desired that can clearly reveal the interaction between these two forms of diversity.

Aiming at obtaining succinct and insightful performance evaluation for MIMO MRT/MRC systems (more general MIMO diversity systems), we take a different approach in this paper by conducting asymptotic analysis. Asymptotic analysis is widely used in various areas of communications and networking. Besides mathematical tractability, asymptotic analysis also helps reveal some fundamental relationship of key system parameters, which may be concealed in the finite case by random fluctuations and other transient properties of channel matrices. This paper comprises two sub-topics: error performance in the single-user scenario and capacity scaling law in the multiuser scenario. While presenting complementary aspects of MIMO MRT/MRC systems, these two are threaded together through a common theme, the investigation of the approximate behavior of the distribution of $\sigma_{\max}^2$ at the extremes, with the former at the origin and the latter at the tail. The main contributions of this paper are summarized below:

1). By studying the behavior of the distribution function of $\sigma_{\max}^2$ at the origin, we obtain the asymptotic average SER (in terms of SNR) for MIMO MRT/MRC systems. As applications we verify the two observations made in [2].

2). By studying the behavior of the distribution function of $\sigma_{\max}^2$ at the tail, we obtain the asymptotic system capacity (in terms of the number of users) for MIMO MRT/MRC systems when multiuser diversity is exploited.

3). Similar analysis is also carried out for two other representative MIMO diversity schemes: STBC/MRC and SC/SC. Comparison among them enables better understanding of MIMO diversity and the interaction between spatial diversity and multiuser diversity.

This paper is organized as follows. In section II, we give our model for MIMO MRT/MRC systems. Then we provide our asymptotic analysis for the average SER and system capacity in Section III and IV respectively, together with some numerical results for illustration purpose. Conclusion is given in Section V.

II. System model

We assume a narrowband MIMO MRT/MRC system with M transmit antennas and N receive antennas, modeled as:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = \mathbf{H}\mathbf{w}_t u + \mathbf{n}, \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^{N\times 1}$ is the received vector, $\mathbf{H} \in \mathbb{C}^{N\times M}$ is the channel matrix, $\mathbf{w}_t \in \mathbb{C}^{M\times 1}$ is a unit-norm transmit weight vector, chosen as the principal right singular vector corresponding to the largest singular value $\sigma_{\max}$ of $\mathbf{H}$, u is the transmitted symbol with power P_T, and $\mathbf{n} \in \mathbb{C}^{N\times 1}$ is a zero-mean circularly symmetric complex Gaussian noise vector with variance $\sigma_n^2/2$ per real dimension. We define $\gamma_t = P_T/\sigma_n^2$ as the average transmit SNR. For illustration purpose, independent and identically distributed Rayleigh fading is considered for $\mathbf{H}$, but our analysis can be readily extended to other fading scenarios when appropriate distributions are available. When multiple MIMO users are involved, their channels are assumed independent. At the receiver side a weight vector $\mathbf{w}_r \in \mathbb{C}^{N\times 1}$ is applied on $\mathbf{y}$ to obtain a decision statistic for u, chosen as the principal left singular vector of $\mathbf{H}$ here. Other diversity schemes can be equivalently represented with $\mathbf{w}_t$ and $\mathbf{w}_r$ appropriately defined.

The cumulative distribution function (CDF) of $\gamma = \sigma_{\max}^2$ is given by [6]

$$F_{\gamma}^{MRT/MRC}(x) = \frac{\left|\mathbf{\Psi}_c(x)\right|}{\Pi_{k=1}^{s}\Gamma(t-k+1)\Gamma(s-k+1)}, \quad x \in (0,+\infty), \tag{2}$$

where $s=\min(M,N)$, $t=\max(M,N)$, and $\mathbf{\Psi}_c(x)$ is an $s\times s$ Hankel matrix function with the (i,j)th entry given by $\{\mathbf{\Psi}_c(x)\}_{i,j} = \gamma(t-s+i+j-1,x)$, for $i,j=1,2,...,s$. Here $\gamma(a,\beta)$ is the incomplete Gamma function defined as $\gamma(a,\beta)=\int_0^{\beta} e^{-t}t^{a-1}dt$, and $\Gamma(a)$ is the Gamma function defined as $\Gamma(a)=\gamma(a,+\infty)$. The probability density function (PDF) of x can be derived as

$$f_{\gamma}^{MRT/MRC}(x) = F_{\gamma}^{MRT/MRC}(x) tr(\mathbf{\Psi}_c^{-1}(x)\mathbf{\Phi}_c(x)), \quad x \in (0,+\infty), \tag{3}$$

where $\mathbf{\Phi}_c(x)$ is an $s\times s$ matrix whose (i,j)th entry is given by $\{\mathbf{\Phi}_c(x)\}_{i,j} = x^{t-s+i+j-2}e^{-x}$.

In the remainder of this paper, we adopt the following notations for the limiting behaviors of two functions $f(x)$ and $g(x)$ with $\lim_{x\to\infty \text{ or } x\to 0} g(x)/f(x)=c$: $g(x)=O(f(x))$ for $0<|c|<\infty$ and specifically $g(x)\sim f(x)$ for $c=1$; $g(x)=o(f(x))$ for $c=0$. When convergence of a sequence of random variables is involved, shorthand notation "*D*" stands for in distribution and "*P*" for in probability.

III. Asymptotic Average SER – Single-User Scenario

In this section, we will derive a succinct expression for average SER at high SNR. The conditional SER for lattice-based modulations can be represented as $P_s(\mathbf{H}) = M_n Q(\sqrt{\kappa\gamma_t\gamma})$, where M_n is the number of the nearest neighboring constellation points, $Q(\cdot)$ is the Gaussian tail *Q*-function, and κ is a positive fixed constant determined by the modulation and coding schemes [5]. At high transmit SNR γ_t, the system average SER $P_s = E\{P_s(\mathbf{H})\}$ will be dominated by the low-probability outage event that γ becomes small [10]. Therefore, only the behavior of $f_{\gamma}^{MRT/MRC}(x)$ at $x\to 0^+$ matters. To this end, the following result is crucial.

Lemma 1: $f_{\gamma}^{MRT/MRC}(x) \sim \dfrac{MN\prod_{k=0}^{s-1}k!}{\prod_{k=0}^{s-1}(t+k)!}x^{MN-1}$, as $x\to 0^+$.

Proof: By Maclaurin Series expansion

$$\{\mathbf{\Psi}_c(x)\}_{i,j} = \gamma(t-s+i+j-1,x) = \frac{1}{t-s+i+j-1}x^{t-s+i+j-1} + o(x^{t-s+i+j-1}), \tag{4}$$

we can obtain the approximation of $\left|\mathbf{\Psi}_c(x)\right|$ at $x=0^+$ after some manipulation as

$$|\mathbf{\Psi}_c(x)| = \left|\mathbf{\Lambda}\right| x^{MN} + o(x^{MN}), \tag{5}$$

with $\{\mathbf{\Lambda}\}_{i,j} = 1/(t-s+i+j-1)$, for $i,j=1,2,...,s$. The determinant of $\mathbf{\Lambda}$ can be obtained in a similar fashion as that of a Hilbert matrix. After some algebra we get

$$\left|\mathbf{\Lambda}\right| = \frac{\prod_{k=0}^{s-1}(k!)^2((t-s+k)!)^2}{\prod_{k=0}^{2s-1}(t-s+k)!}, \tag{6}$$

and it follows from (2) that

$$F_{\gamma}^{MRT/MRC}(x)=\frac{\prod_{k=0}^{s-1}k!}{\prod_{k=0}^{s-1}(t+k)!}x^{MN}+o(x^{MN})\,. \tag{7}$$

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With Lemma 1, we establish the following result for the asymptotic average SER for MIMO MRT/MRC systems following Proposition I in [10].

Proposition 1: For MIMO MRT/MRC systems, the asymptotic average SER is given by

$$P_s=\frac{2^{q^{(MRT/MRC)}}M_n\alpha^{(MRT/MRC)}\Gamma(q^{(MRT/MRC)}+\frac{3}{2})}{\sqrt{\pi}(q^{(MRT/MRC)}+1)}\times(\kappa\gamma_t)^{-(q^{(MRT/MRC)}+1)}+o(\gamma_t^{-(q^{(MRT/MRC)}+1)}). \tag{8}$$

where

$$\alpha^{(MRT/MRC)}=\frac{MN\prod_{k=0}^{s-1}k!}{\prod_{k=0}^{s-1}(t+k)!},\ \ q^{(MRT/MRC)}=MN-1. \tag{9}$$

The validity of (8) is demonstrated in Fig. 1 for uncoded BPSK systems. Based on (8), one readily concludes that the optimal diversity order for MIMO diversity systems is $M\times N$. Therefore, if we keep $M+N$ fixed (a measure of system cost), even distribution of the number of transmit and receive antennas (more precisely a smallest $|M-N|$) maximizes $M\times N$, thus minimizing the system SER at high SNR. On the other hand, when comparing two MIMO MRT/MRC systems with the same diversity order $M\times N$, the one with smaller $\alpha^{(MRT/MRC)}$ yields larger coding gain and thus smaller SER (in this case, $q^{(MRT/MRC)}$ is a constant). We can conclude that in this scenario, the sum of transmit and receive antennas should be made as large as possible, with the optimum achieved at $s=1$ and $t=M\times N$. This conclusion is based on the following result regarding $\alpha^{(MRT/MRC)}$ as a function of M and N (or equivalently of s and t).

Lemma 2: Given four positive integers s_1, t_1, s_2, t_2, assume $s_1\times t_1=s_2\times t_2$, $s_1<t_1$, $s_2<t_2$, and $s_1+t_1>s_2+t_2$, then $\alpha^{(MRT/MRC)}(s_1,t_1)<\alpha^{(MRT/MRC)}(s_2,t_2)$.

Proof: From $s_1+t_1>s_2+t_2$, we can obtain $s_1<s_2<t_2<t_1$. As

$$\alpha^{(MRT/MRC)}(s_1,t_1)=\frac{\prod_{k=0}^{s_1-1}k!}{\prod_{k=0}^{s_1-1}(t_1+k)!}=\frac{1}{1\times2\times\ldots\times t_1}\bullet\frac{1}{2\times3\times\ldots\times(t_1+1)}\bullet\ldots\frac{1}{s_1\times\ldots\times(s_1+t_1-1)}, \tag{10}$$

$$\alpha^{(MRT/MRC)}(s_2,t_2)=\frac{\prod_{k=0}^{s_2-1}k!}{\prod_{k=0}^{s_2-1}(t_2+k)!}=\frac{1}{1\times2\times\ldots\times t_2}\bullet\frac{1}{2\times3\times\ldots\times(t_2+1)}\bullet\ldots\frac{1}{s_2\times\ldots\times(s_2+t_2-1)}, \tag{11}$$

it is equivalent to show that

$$(1\times\ldots\times t_1)\times\ldots\times(s_1\times\ldots\times(s_1+t_1-1))>(1\times\ldots\times t_2)\times\ldots\times(s_2\times\ldots\times(s_2+t_2-1)). \tag{12}$$

The left hand side of (12) can be rewritten as

$$1^{f(1)}\times2^{f(2)}\times\ldots\times(s_1+t_1-1)^{f(s_1+t_1-1)}, \tag{13}$$

with

$$f(i)=\begin{cases} i, & 1\le i\le s_1 \\ s_1, & s_1+1\le i\le t_1 \\ s_1+t_1-i, & t_1+1\le i\le s_1+t_1-1. \end{cases} \tag{14}$$

Similarly the right hand side of (12) can be represented as

$$1^{g(1)}\times 2^{g(2)}\times ...\times (s_2+t_2-1)^{g(s_2+t_2-1)}, \tag{15}$$

with

$$g(i)=\begin{cases} i, & 1\le i\le s_2 \\ s_2, & s_2+1\le i\le t_2 \\ s_2+t_2-i, & t_2+1\le i\le s_2+t_2-1. \end{cases} \tag{16}$$

It is not difficult to get $\sum_{i=1}^{s_1+t_1-1} f(i)=s_1\times t_1=s_2\times t_2=\sum_{i=1}^{s_2+t_2-1} g(i)$. Therefore, after canceling out the same factors in (13) and (15), we can see that (13) is surely larger than (15).

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From the asymptotic SER expression in (8), we have verified the two observations made in [2] rigorously at high SNR. Below we will follow a similar approach to compute the corresponding parameters for the coding gain and diversity order for MIMO STBC/MRC and SC/SC systems (whose asymptotic average SERs assume the same forms as (8)).

Without loss of generality, we assume that the adopted space-time block coding scheme achieves the full rate and the transmit power is equally allocated among the transmit antennas. In this case, the normalized effective link SNR for a generic user is given by $\gamma=\frac{1}{M}\sum_{i=1}^{N}\sum_{j=1}^{M}\left|h_{i,j}\right|^2$, whose PDF admits:

$$f_\gamma^{STBC/MRC}(x)=\frac{M^{MN}}{(MN-1)!}x^{MN-1}e^{-Mx},\quad x\ge 0. \tag{17}$$

Similarly the corresponding parameters for the coding gain and diversity order for MIMO STBC/MRC systems can be obtained as

$$\alpha^{(STBC/MRC)}=\frac{M^{MN}}{(MN-1)!},\quad q^{(STBC/MRC)}=MN-1. \tag{18}$$

For the SC/SC scheme, both the user and the base station choose one optimal antenna such that the resultant channel gain is maximized. Thus the normalized effective link SNR at the receiver is $\gamma=\max_{1\le i\le N,1\le j\le M}\left(\left|h_{i,j}\right|^2\right)$, whose PDF can be easily obtained as

$$f_\gamma^{SC/SC}(x)=MNe^{-x}(1-e^{-x})^{MN-1},\quad x\ge 0. \tag{19}$$

We can obtain the corresponding parameters for the coding gain and diversity order for MIMO SC/SC systems as

$$\alpha^{(SC/SC)}=MN,\quad q^{(SC/SC)}=MN-1. \tag{20}$$

Comparing (9), (18) and (20) we can see that all these MIMO diversity schemes achieve the same diversity order. Nonetheless, their error performances could still be dramatically different owing to different coding gains, as exhibited in Fig. 2. For example, when $M=6$ and $N=1$, our asymptotic results predict a SNR gap of 4.7 dB between MRT/MRC ($\alpha^{(MRT/MRC)}=1/120$) and SC/SC ($\alpha^{(SC/SC)}=6$), and 7.8 dB between MRT/MRC and STBC/MRC ($\alpha^{(STBC/MRC)}=388.8$) for uncoded BPSK systems at high SNR, which agree well with simulation results (see Fig.3 at SER 10^{-5}). It is also observed that for the same diversity order, the performance of STBC/MRC worsens with the increase of the number of transmit antennas.

IV. Asymptotic System Capacity - Multiuser Scenario

In this section, we consider a homogeneous downlink multiuser MIMO communication scenario, which is envisioned to be of crucial importance for emerging wireless networks. We will explore how the average (ergodic) system capacity of a multiuser MIMO MRT/MRC system scales with the number of users *K* when opportunistic scheduling is employed, and how the number of antennas *M* and *N* come into play. Assume the normalized effective *link SNR* for user k is γ_k, whose PDF and CDF are denoted by $f_\gamma(x)$ and $F_\gamma(x)$ respectively (same for all users). In the opportunistic scheduling scheme, the base station chooses the user $k^*=\arg\max_k(\gamma_k)_{k=1}^K$. Thus the resultant normalized *system SNR* seen by the base station is γ_{k^*} with PDF

$$f_{\gamma_{k^*}}(x)=Kf_\gamma(x)F_\gamma^{K-1}(x). \tag{21}$$

Assuming that average transmit SNR is γ_t, average system capacity obtained by opportunistic scheduling can be expressed as

$$E\left(\log\left(1+\gamma_t\left(\max_{1\le k\le K}\gamma_k\right)\right)\right)=\int_0^{+\infty}\log\left(1+\gamma_t x\right)f_{\gamma_{k^*}}(x)dx. \tag{22}$$

The closed-form expression for (22) is rather complicated, especially for MIMO MRT/MRC systems. We therefore resort to the theory of order statistics for asymptotic analysis [11][12]. Some related pioneer study on spatial multiplexing systems can be found in [13]. To this end, the tail behavior of $f_\gamma^{MRT/MRC}(x)$ is required, which we state below.

Lemma 3: $f_\gamma^{MRT/MRC}(x)\sim\dfrac{1}{(M-1)!(N-1)!}e^{-x}x^{M+N-2}$, as $x\to+\infty$.

Proof: When $x\to+\infty$, $F_\gamma^{MRT/MRC}(x)\to 1$, and

$$\lim_{x\to\infty}\{\mathbf{\Psi}_c(x)\}_{i,j}=\lim_{x\to\infty}\gamma(t-s+i+j-1,x)=(t-s+i+j-2)!. \tag{23}$$

Assume $\lambda=t-s$, then $\mathbf{\Psi}_c(+\infty)$ is given by

$$\mathbf{\Psi}_c(+\infty)=\begin{bmatrix}\lambda! & (\lambda+1)! & .. & (\lambda+s-1)!\\ (\lambda+1)! & & & \\ \vdots & \vdots & \vdots & \vdots\\ (\lambda+s-1)! & .. & .. & (\lambda+2s-2)!\end{bmatrix}_{s\times s}. \tag{24}$$

Since

$$\boldsymbol{\Phi}_c(x)=\begin{bmatrix} x^{\lambda}e^{-x} & x^{\lambda+1}e^{-x} & .. & x^{\lambda+s-1}e^{-x} \\ x^{\lambda+1}e^{-x} & : & : & : \\ : & : & : & : \\ x^{\lambda+s+1}e^{-x} & : & : & x^{\lambda+2s-2}e^{-x} \end{bmatrix}=\begin{bmatrix} 1 & x & .. & x^{s-1} \\ x & : & : & : \\ : & : & : & : \\ x^{s-1} & : & : & x^{2s-2} \end{bmatrix}x^{\lambda}e^{-x}, \tag{25}$$

the tail behavior of $f_{\gamma}^{MRT/MRC}(x)$ will be determined by that of $\boldsymbol{\Phi}_c(x)$, given by (where the coefficients $\{a_i\}$ come from linear combinations of elements in $\boldsymbol{\Psi}_c^{-1}(+\infty)$)

$$\begin{aligned} f_{\gamma}^{MRT/MRC}(x) &\sim tr(\boldsymbol{\Psi}_c^{-1}(+\infty)\boldsymbol{\Phi}_c(x)) = e^{-x}[a_1x^{\lambda+2s-2}+a_2x^{\lambda+2s-3}+...+a_{2s-2}x^{\lambda+1}+a_{2s-1}x^{\lambda}] \\ &= e^{-x}x^{\lambda+2s-2}[a_1+O(1/x)], \end{aligned} \tag{26}$$

with

$$a_1=\frac{\left|\begin{pmatrix} \lambda! & (\lambda+1)! & ... & (\lambda+s-2)! \\ (\lambda+1)! & .. & .. & (\lambda+s-1)! \\ : & : & : & : \\ (\lambda+s-2)! & .. & .. & (\lambda+2s-4)! \end{pmatrix}\right|}{\left|\boldsymbol{\Psi}_c(+\infty)\right|}=\frac{\prod_{k=1}^{s-1}(t-k-1)!(s-k-1)!}{\prod_{k=1}^{s}(t-k)!(s-k)!} \tag{27}$$

$$=\frac{1}{(t-1)!(s-1)!}=\frac{1}{(M-1)!(N-1)!}.$$

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With Lemma 3, we derive the asymptotic system capacity for multiuser MIMO MRT/MRC systems as follows.

Proposition 2: When multiuser diversity is exploited in a *K*-user MIMO MRT/MRC system, the asymptotic average system capacity $\overline{C}_K^{(MRT/MRC)}$ is given by

$$\overline{C}_K^{(MRT/MRC)}=E\left(\log\left(1+\gamma_t\left(\max_{1\le k\le K}\gamma_k^{(MRT/MRC)}\right)\right)\right)\to\log\left(1+\gamma_t b_K^{(MRT/MRC)}\right),\ \text{ as } K\to\infty, \tag{28}$$

where $b_K^{(MRT/MRC)}$ is solved through $F_{\gamma}^{(MRT/MRC)}(b_K)=1-1/K$ and is given by

$$b_K^{(MRT/MRC)}=\log\left(\frac{K}{(M-1)!(N-1)!}\right)+(M+N-2)\log\log\left(\frac{K}{(M-1)!(N-1)!}\right)+O(\log\log\log K). \tag{29}$$

Proof: See Appendix.

Remark: The following result is often invoked to indicate that $\max_{1\le k\le K}\gamma_k$ "grows like" b_K in a coarse sense, and is widely used in the study of opportunistic communications involving extreme values and order statistics (e.g., [7][14]):

$$\frac{\max_{1\le k\le K}\gamma_k-b_K}{a_K}\xrightarrow{D}\Lambda(x)=\exp\left(-e^{-x}\right), \tag{30}$$

where $a_K = (K f_\gamma(b_K))^{-1}$. This result can actually be strengthened from existing literature [11][12]: if $\lim_{x\to\infty} \frac{1-F_\gamma(x)}{f_\gamma(x)} = c = 0$, $\max_{1\le k\le K} \gamma_k - b_K \xrightarrow{P} 0$, otherwise if $\lim_{x\to\infty} \frac{1-F_\gamma(x)}{f_\gamma(x)} = c > 0$ $\max_{1\le k\le K} \gamma_k / b_K \xrightarrow{P} 1$. Nonetheless, our result (28) is a yet stronger one, which is concerned with the convergence of the expected values of functions of $\max_{1\le k\le K} \gamma_k$.

In a similar fashion, we can obtain the asymptotic system capacity for multiuser MIMO STBC/MRC and SC/ SC systems, which are dictated by

$$\overline{C}_K^{(STBC/MRC)} \to \log\left[1+\gamma_t\left(\frac{1}{M}\log cK + (N-\frac{1}{M})\log\log\left(cK\right) + O(\log\log\log K)\right)\right], \tag{31}$$

where

$$c = \frac{M^{MN-1}}{(MN-1)!}, \tag{32}$$

and[1]

$$\overline{C}_K^{(SC/SC)} \to \log\left[1+\gamma_t\left(\log MNK\right)\right], \tag{33}$$

as $K\to\infty$.

From the asymptotic system capacities of the joint spatial diversity and multiuser diversity systems, we can make some interesting observations. From (31), a tradeoff between transmit diversity and multiuser diversity for an open-loop spatial diversity system is seen, which has also been observed by other researchers (e.g., [14][15]). But in our paper, a more rigorous proof is provided and how the asymptotic system capacity is related to key system parameters is revealed. For example, our result does show the positive role of the number of receive antennas *N*, though in a second-order[2] sense, which is not clear from previous results in literature. It is also observed that the detrimental effect of multiple transmit antennas can be avoided with the closed-loop spatial diversity schemes, as seen in (29) and (33)[3]. Also from (29) and (33), we can infer that for the general hybrid selection combining schemes, the scaling laws should only have differences in the second order approximations. Numerical results in Fig. 4 verify that $\log\left(1+\gamma_t b_K\right)$ is a good approximation for the average capacity of the STBC/MRC, SC/SC and MRT/MRC systems using the opportunistic scheduler.

V. Conclusions

In this paper, through the analysis of the distribution of the squared largest singular value of a

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[1] This is a rare accurate expression. Note that in this case, the growth in transmit and receive antennas can be equivalently seen as an increase in the number of users (due to the i.i.d. assumptions).

[2] We define the first order approximation when truncated at $\log K$, and the second order approximation when truncated at $\log\log K$.

[3] The coefficient of K is not important when K becomes large. In this sense, multiple antennas even help for the MRT/MRC scheme.

complex Gaussian matrix at the origin and tail, we obtain two asymptotic results related to MIMO MRT/MRC systems. One is the asymptotic error performance in the single-user scenario at high transmit SNR, and the other is the asymptotic system capacity in the multiuser scenario when multiuser diversity is exploited. Our results are rigorous and succinct, which provide a performance reference for MIMO diversity systems and facilitate various tradeoff studies in terms of system parameters and designs.

Appendix: Proof of Proposition 2

Proof: For the purpose of brevity, we will use $F(x)$ and $f(x)$ to denote $F_{\gamma}^{MRT/MRC}(x)$ and $f_{\gamma}^{MRT/MRC}(x)$ respectively in the following proof, and b_K for $b_K^{(MRT/MRC)}$. Assume $C_K = \log\left(1+\gamma_t\left(\max_{1\le k\le K}\gamma_k\right)\right)$. Define the growth function $g(x) = (1-F(x))/f(x)$, with Lemma 3 we have

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}\frac{-f(x)}{f'(x)} = 1. \tag{34}$$

Clearly $F(x)$ in (2) is less than 1 for all finite x and is twice differentiable for all x. By (19) of [16], we can obtain the following expansion at b_K

$$\begin{aligned}\log\left[-\log F^K(b_K + xg(b_K))\right] &= -x + \frac{x^2}{2!}g'(b_K) + \frac{x^3}{3!}\left[g(b_K)g^{(2)}(b_K) - 2g'^2(b_K)\right]... + ...\\ &+\frac{e^{-x}+...}{2K} + \frac{5e^{-2x+...}}{24K^2} + ... - \frac{1}{8K^3}e^{-3x} + ... + ...,\end{aligned} \tag{35}$$

where b_K is given by $F(b_K) = 1 - 1/K$. Solving for b_K we can get (for some constant c_1)

$$b_K = \log c_1 K + (M+N-2)\log\log c_1 K + O(\log\log\log K) = O(\log K). \tag{36}$$

A close examination of $g'(x)$ using Lemma 3 reveals

$$g'(x) = O(1/x), \text{and } \lim_{K\to\infty}\left[Kg'(b_K)\right] = +\infty. \tag{37}$$

Therefore, the terms in the second line of (35) starting with the term $e^{-x}/2K$ can be ignored [16]. Further exploiting (34), (36) and (37) in the first line of (35) with $x = \pm\log\log K$ yields[4]

$$\Pr\left\{-\log\log K \le \left(\max_{1\le k\le K}\gamma_k\right) - b_K \le \log\log K\right\} \ge 1 - O\left(\frac{1}{\log K}\right). \tag{38}$$

Apply Chebyshov's inequality, we have

$$E\left(C_K\right) \ge P\left(C_K \ge \log\left(1+\gamma_t\left(b_K - \log\log K\right)\right)\right)\times\log\left(1+\gamma_t\left(b_K - \log\log K\right)\right)$$

---

[4] It can be shown that (38) still holds for a more general condition $g'(x) = O(1/x^{\delta})$ with $\delta > 0$.

$$\geq\left(1-O\left(\frac{1}{\log K}\right)\right)\times\log\left(1+\gamma_t\left(b_K-\log\log K\right)\right)$$

$$=\log\left(1+\gamma_t\left(b_K-\log\log K\right)\right)-O\left(\frac{\log\log K}{\log K}\right) \quad (39)$$

$$=\log\left(1+\gamma_t b_K\right)-o(1).$$

On the other hand,

$$E\left(C_K\right)=\int_0^{\infty}P\left(C_K>x\right)dx$$

$$=\int_0^{\log(1+\gamma_t b_K)}P\left(C_K>x\right)dx+\int_{\log(1+\gamma_t b_K)}^{+\infty}P\left(C_K>x\right))dx \quad (40)$$

$$\leq\log\left(1+\gamma_t b_K\right)+\int_{\log(1+\gamma_t b_K)}^{+\infty}P\left(C_K>x\right)dx,$$

with

$$P(C_K>x)=1-P\left(C_K\leq x\right)=1-F^K\left(\frac{e^x-1}{\gamma_t}\right). \quad (41)$$

We know $\lim_{x\to\infty}\frac{1-F(x)}{f(x)}=1>0$, therefore when x is large enough, we can find a positive constant c_2 and x_0, such that $1-F(x)<c_2 f(x)$, for any $x>x_0$. Thus for sufficiently large x

$$1-F^K\left(\frac{e^x-1}{\gamma_t}\right)=\left(1-F\left(\frac{e^x-1}{\gamma_t}\right)\right)\left(1+F\left(\frac{e^x-1}{\gamma_t}\right)+\ldots+F^{(K-1)}\left(\frac{e^x-1}{\gamma_t}\right)\right)\leq Kc_2 f\left(\frac{e^x-1}{\gamma_t}\right). \quad (42)$$

Therefore when K is large enough, we have

$$\int_{\log(1+\gamma_t b_K)}^{+\infty}P(C_K>x)dx\leq\int_{\log(1+\gamma_t b_K)}^{+\infty}Kc_2 f\left(\frac{e^x-1}{\gamma_t}\right)dx$$

$$=\int_{b_K}^{+\infty}Kc_2 f(x)\frac{\gamma_t}{1+x\gamma_t}dx\leq\frac{c_2\gamma_t}{1+\gamma_t b_K}\int_{b_K}^{+\infty}Kf(x)dx$$

$$=O\left(\frac{1}{\log K}\right)\times\int_{b_K}^{+\infty}Kf\left(x\right)dx=O\left(\frac{1}{\log K}\right)\times K\times\left(1-F\left(b_K\right)\right) \quad (43)$$

$$=O\left(\frac{1}{\log K}\right),$$

where the last equality uses the fact $\left(1-F\left(b_K\right)\right)=1/K$. So for sufficiently large K

$$E\left(C_K\right)\leq\log\left(1+\gamma_t b_K\right)+O\left(\frac{1}{\log K}\right). \quad (44)$$

Based on (39) and (44) we can conclude

$$\lim_{K\to\infty}\left\{E\left(\log\left(1+\gamma_t\left(\max_{1\leq k\leq K}\gamma_k\right)\right)\right)-\log\left(1+\gamma_t b_K\right)\right\}\to 0. \quad (45)$$

■

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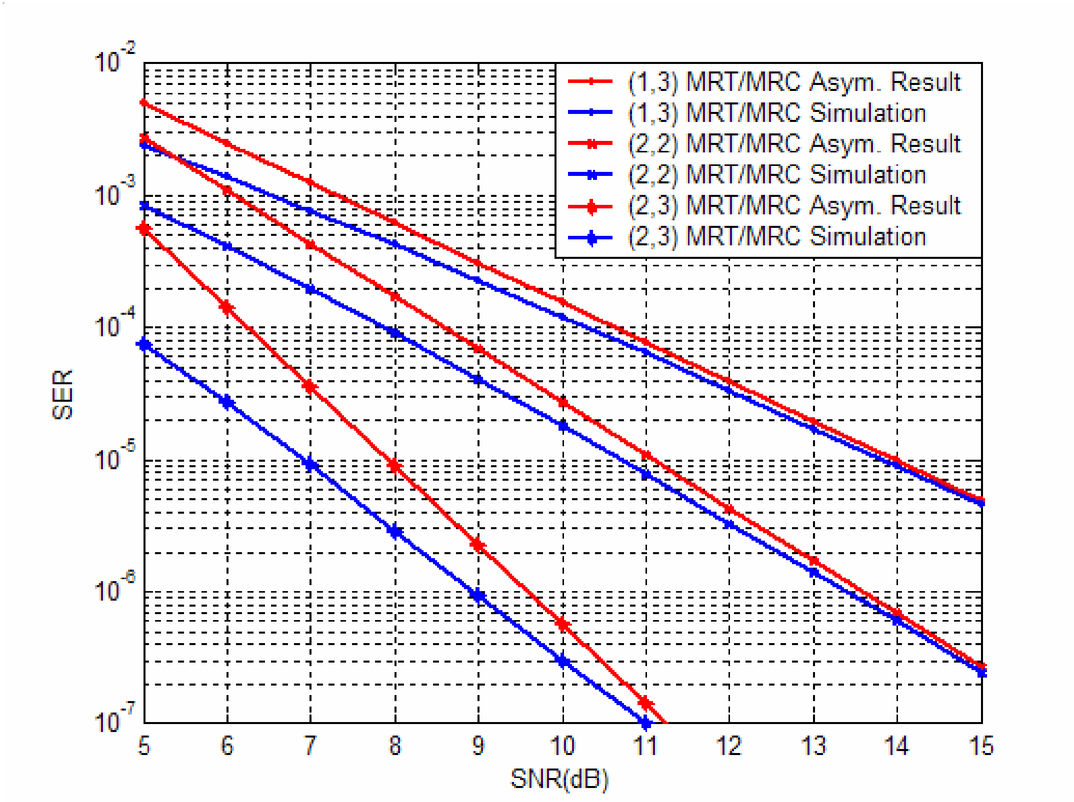

Fig.1 Comparison between asymptotic and simulation results for BPSK under different antennas configurations (the notation (M, N) refers to MIMO systems with M transmit and N receive antennas)

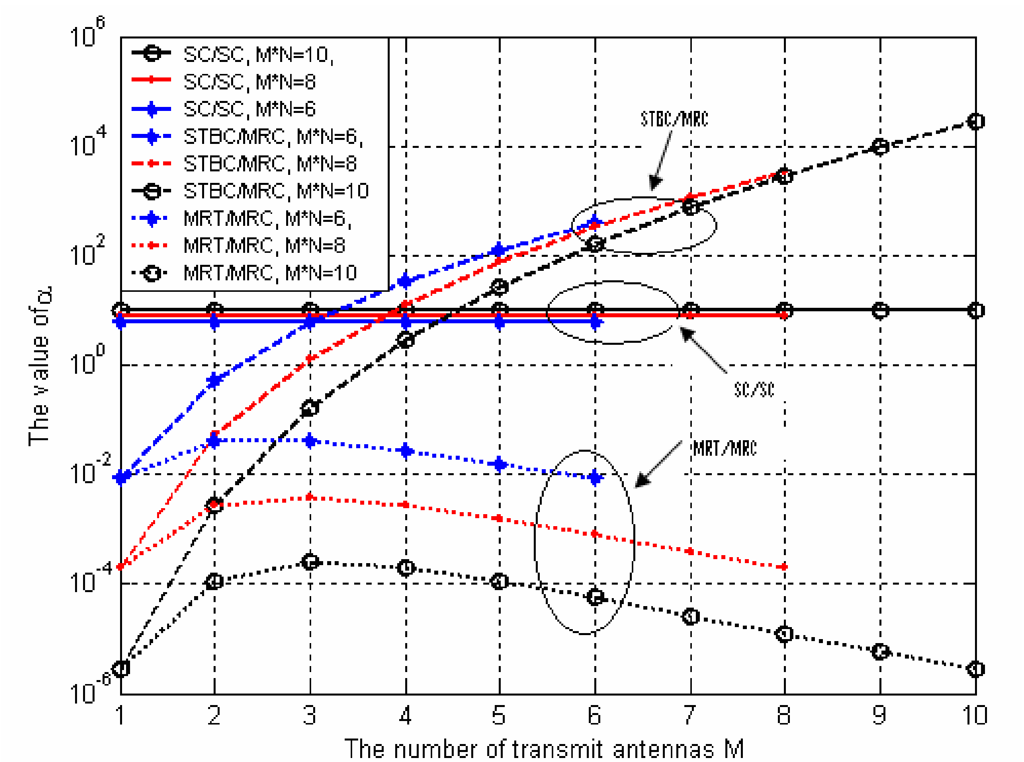

Fig.2 Coding gain parameter α with the number of transmit antennas for the same diversity order $M \times N$

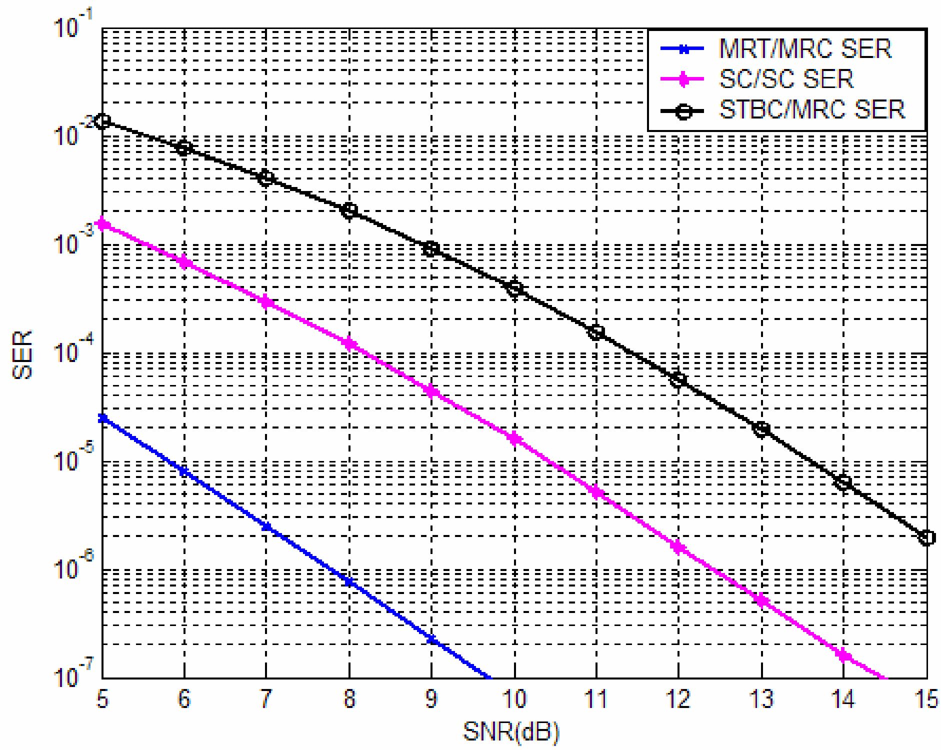

Fig.3 Symbol error rate of the three MIMO diversity schemes for BPSK ($M = 6, N = 1$)

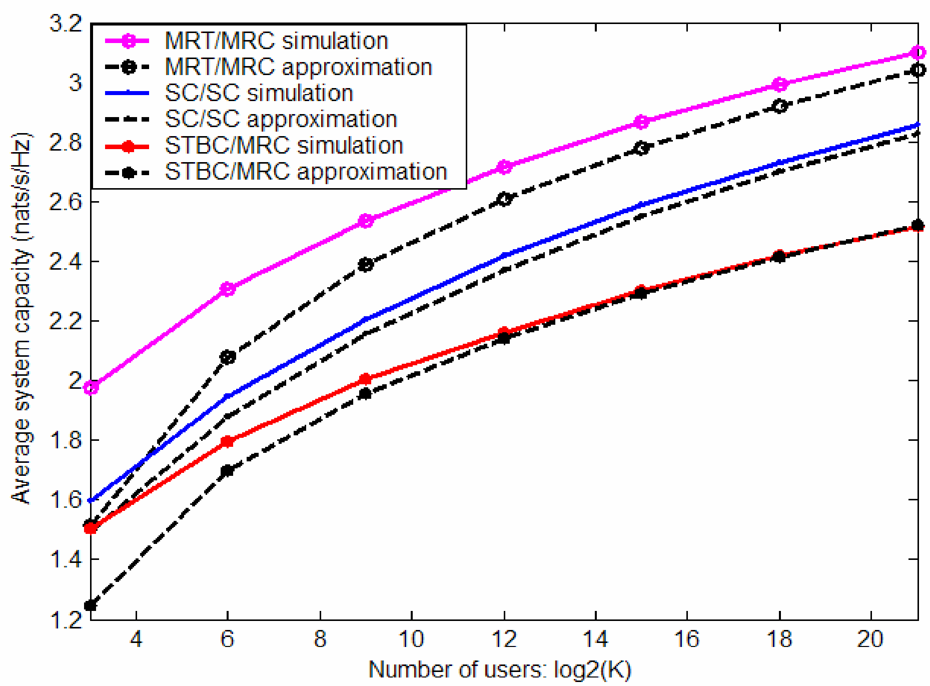

Fig.4 Average system capacity of opportunistic scheduling ($\gamma_t = 0\ dB,\ M = N = 2$)