

THE IMPACT OF SHADOW FADING ON THE OUTAGE CAPACITY AND MULTIUSER SCHEDULING GAIN OF MIMO SYSTEMS

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ABSTRACT

Shadow fading is generally not explicitly studied for MIMO systems. In this paper, through asymptotic large-system analysis, it is shown that shadow fading, while significantly limits the channel outage capacity as expected, actually enhances the multiuser scheduling gain, which is meaningful and obtainable for delay-tolerant applications.

1. INTRODUCTION

Current study of multi-input multi-output (MIMO) systems seldom explicitly addresses the shadow fading issues, though it is natural to expect severely diminished link quality when unfavorable shadowing is experienced [1][2]. While fading generally increases the dynamism of individual link quantities, which leads to larger outage probability and is unfavorable to real-time applications, it can actually contribute to the scheduling gain in a multiuser environment for delay-tolerant applications [3][4]. This paper intends to quantify the effect of shadow fading on individual link outage capacity and multiuser scheduling gain for MIMO systems, and reveal certain tradeoff between multiple antennas and multiuser diversity.

This paper is organized as follows. Section II presents the system model. In Section III, the effect of shadowing on MIMO outage capacity is explicitly quantified, while the corresponding scaling law for multiuser scheduling gain is derived in Section IV. Finally, Section V contains some concluding remarks.

2. SYSTEM MODEL

Consider a multiuser MIMO system, with M antennas at the base station and N antennas at each of the K users. In this paper we mainly study the downlink scenario, but all

discussions can be extended to the uplink as well. Each link is modeled as (user index is omitted for simplicity)

$$\mathbf{y} = \Phi^{1/2} \mathbf{H} \mathbf{x} + \mathbf{n}, \quad (1)$$

where \mathbf{y} is the received vector, \mathbf{x} is the transmitted vector with the total transmitted power ρ equally divided among transmit antennas, \mathbf{H} is the channel matrix representing small-scale fading while Φ captures the common large-scale fading effect, and \mathbf{n} is the noise vector. The entries of \mathbf{H} and \mathbf{n} are independent and identically distributed (i.i.d.) complex Gaussian with zero mean and unit variance. Note that $\Phi = 1$ represents the common pure Rayleigh fading model. We assume a block fading scenario where \mathbf{H} changes independently from one block to another, and Φ changes independently with \mathbf{H} at a lower pace.

Without loss of generality, we exclude the path loss effect and assume i.i.d shadow fading for different users. The shadow fading coefficient is commonly modeled as $\Phi = e^Y$, where $Y \sim \mathcal{N}(\lambda\mu_L, (\lambda\sigma_L)^2)$ is a Gaussian random variable, with μ_L (dB) the area mean, and σ_L (dB) the decibel spread, typically ranging between 6-12 depending on the severity of the shadow fading, and $\lambda = \ln 10 / 10$. The cumulative distribution function (CDF) of Φ is given by

$$F_\Phi(x) = 1 - Q\left(\frac{\ln x - \lambda\mu_L}{\lambda\sigma_L}\right), \quad (2)$$

where $Q(\cdot)$ is the standard Gaussian tail function. We also have

$$E\{\Phi\} = e^{\lambda\mu_L + \frac{\lambda^2\sigma_L^2}{2}}, \quad E\{\Phi^2\} = e^{2\lambda\mu_L + 2\lambda^2\sigma_L^2}. \quad (3)$$

Our study focuses on two asymptotic scenarios: (1) large M and fixed N and (2) large M and N with their ratio fixed. Besides analytical tractability through laws of large numbers and random matrix theory, the study of large system performance also has practical advantages: what is revealed in the asymptotic limit is fundamental in nature, which may be concealed in the finite case by random fluctuations and other transient properties of the matrix entries; moreover, the convergence to the asymptotic limit

is typically rather fast as the system size grows [5]-[7]. We also focus our study on the high SNR regimes.

In the following, when convergence of a sequence of random variables is involved, shorthand notation “D” stands for in distribution, “P” for in probability, and “a.s.” for almost surely. “log” is used for logarithm with an arbitrary base, and “ln” for base e .

3. OUTAGE CAPACITY

Outage capacity is a common metric in MIMO communications with block fading assumptions, where the instantaneous channel capacity

$$C = \log \det \left(\mathbf{I} + \frac{\rho}{M} \Phi \mathbf{H} \mathbf{H}^H \right) \quad (4)$$

is a random quantity varying with \mathbf{H} and Φ . An outage capacity $C^{(p)}$ is defined with respect to an outage probability $p = P(C < C^{(p)})$. Outage capacity is especially meaningful for delay-constrained applications where the data rate cannot be adapted with channel variations. Our main results on the impact of shadow fading are given below.

Proposition 1: For large M , ρ and fixed N , and log-normal shadowing,

$$C^{(p)} = N(\log \rho + \lambda \mu_L \log e) - \log e \sqrt{\frac{N}{M} + N^2 \lambda^2 \sigma_L^2} Q^{-1}(p). \quad (5)$$

Proof: Following a similar approach as Theorem 2 in [5], it can be shown that, conditioned on Φ , for large M and fixed N

$$\sqrt{\frac{M}{N}} \frac{(1 + \rho \Phi)}{\rho \Phi \log e} [C - N \log(1 + \rho \Phi)] \xrightarrow{D} \mathcal{N}(0, 1), \quad (6)$$

which indicates that at high SNR

$$C \sim \mathcal{N} \left(N \log(\rho \Phi), \frac{N}{M} \log^2 e \right). \quad (7)$$

Therefore,

$$p = E_{\Phi} \left\{ Q \left(\frac{N(\log \rho + \log \Phi) - C^{(p)}}{\log e \sqrt{N/M}} \right) \right\}. \quad (8)$$

For log-normal shadowing, $\log \Phi \sim \mathcal{N}(\lambda \mu_L, (\lambda \sigma_L)^2)$, so we can further simplify (8) as

$$p = Q \left(\frac{N(\log \rho + \lambda \mu_L \log e) - C^{(p)}}{\log e \sqrt{\frac{N}{M} + N^2 \lambda^2 \sigma_L^2}} \right), \quad (9)$$

and (5) follows. ■

Remark: For pure Rayleigh fading, it is obtained in [5] that

$$C^{(p)} = N \log \rho - \log e \sqrt{\frac{N}{M}} Q^{-1}(p). \quad (10)$$

For large M , ρ and fixed N . Note that there is a non-vanishing component in the coefficient of $Q^{-1}(p)$ in (5) when $M \rightarrow \infty$, verifying the detrimental effect of shadowing fading on the MIMO outage capacity.

Similar results can be obtained when both M and N are large while their ratio keeps fixed.

Proposition 2: For large M , N , ρ , with $M/N \rightarrow \beta$, and log-normal shadowing,

$$C^{(p)} = \begin{cases} M \left[\left(\log \frac{\rho}{e} + \log \frac{1-\beta}{\beta} + \frac{1}{\beta} \log \frac{1}{1-\beta} \right) + \lambda \mu_L \log e \right] \\ - \log e \sqrt{\ln \frac{1}{1-\beta} + M^2 \lambda^2 \sigma_L^2} Q^{-1}(p), & \beta < 1 \\ N \left[\log \frac{\rho}{e} + \lambda \mu_L \log e \right] \\ - \log e \sqrt{(\ln N + \gamma + 1) + N^2 \lambda^2 \sigma_L^2} Q^{-1}(p), & \beta = 1 \\ N \left[\left(\log \frac{\rho}{e} + (\beta - 1) \log \frac{\beta}{\beta - 1} \right) + \lambda \mu_L \log e \right] \\ - \log e \sqrt{\ln \frac{\beta}{\beta - 1} + N^2 \lambda^2 \sigma_L^2} Q^{-1}(p), & \beta > 1. \end{cases} \quad (11)$$

Remark: γ is the Euler constant. The proof follows a similar argument as in Proposition 1, after showing the asymptotic Gaussianity of the conditional instantaneous channel capacity as in [9] and [5]. This proposition should be compared with Corollary 1 in [5]. Again, shadow fading worsens the outage capacity (reflected as terms containing λ above).

4. MULTIUSER SCHEDULING GAIN

For delay-tolerant applications, multiuser diversity can be exploited to enhance the system throughput by always scheduling the user with the best link quality for communication [3][4]. To establish our main results on the impact of shadow fading, we need the following lemma.

Lemma 1: If Z_1, \dots, Z_K are i.i.d. and, conditioned on some random variable θ , are Gaussian distributed with the conditional mean $\mu(\theta)$ and the conditional variance σ^2 , which does not depend on θ , $\max_{1 \leq k \leq K} Z_k$ converges as

$K \rightarrow \infty$ in probability to b_K , the solution to

$$E_{\theta} \left\{ Q \left(\frac{b_K - \mu(\theta)}{\sigma} \right) \right\} = \frac{1}{K}. \quad (12)$$

Proof: We define the CDF and PDF of $\{Z_k\}$ as

$$F_Z(x) = E_\theta\{F_G(x;\theta)\} \text{ and } f_Z(x) = E_\theta\{f_G(x;\theta)\}, \quad (13)$$

where $F_G(x;\theta)$ and $f_G(x;\theta)$ are CDF and PDF of a Gaussian random variable with mean $\mu(\theta)$ and variance σ^2 . Then as $x \rightarrow \infty$, the following approximation becomes accurate:

$$1 - F_G(x;\theta) \approx \frac{\sigma^2}{x} f_G(x;\theta) \text{ and } -f_G'(x;\theta) \approx \frac{x}{\sigma^2} f_G(x;\theta). \quad (14)$$

One can readily show that

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{d}{dx} \left[\frac{1 - F_Z(x)}{f_Z(x)} \right] \\ &= -1 + \lim_{x \rightarrow \infty} \frac{-(1 - F_Z(x))f_Z'(x)}{f_Z^2(x)} \\ &= -1 + \lim_{x \rightarrow \infty} \frac{-(1 - E_\theta\{F_G(x;\theta)\})E_\theta\{f_G'(x;\theta)\}}{[E_\theta\{f_G(x;\theta)\}]^2} \quad (15) \\ &= -1 + \lim_{x \rightarrow \infty} \frac{E_\theta\left\{\frac{\sigma^2}{x} f_G(x;\theta)\right\} E_\theta\left\{\frac{x}{\sigma^2} f_G(x;\theta)\right\}}{[E_\theta\{f_G(x;\theta)\}]^2} \\ &= 0, \end{aligned}$$

where we have assumed that derivation and expectation can be exchanged under some mild conditions. By Theorem 10.5.2 in [10], the standardized extreme $\left(\max_{1 \leq k \leq K} Z_k - b_K\right)/a_K$ has a limiting distribution $G_3(x) = \exp(-e^{-x})$, where

$$b_K = F_Z^{-1}(1 - 1/K) \text{ and } a_K = (Kf_Z(b_K))^{-1}. \quad (16)$$

It can be further shown that

$$\begin{aligned} \lim_{K \rightarrow \infty} \frac{1}{a_K} &= \lim_{K \rightarrow \infty} Kf_Z(b_K) = \lim_{K \rightarrow \infty} \frac{f_Z(b_K)}{1 - F_Z(b_K)} \\ &= \infty, \end{aligned} \quad (17)$$

so $\forall \varepsilon > 0$,

$$\lim_{K \rightarrow \infty} P\left(\left|\max_{1 \leq k \leq K} Z_k - b_K\right| > \varepsilon\right) = 1 - \lim_{K \rightarrow \infty} G_3(\varepsilon/a_K) = 0. \quad (18)$$

Therefore, $\max_{1 \leq k \leq K} Z_k \xrightarrow{P} b_K$. ■

It is known from Proposition I that For large M , ρ and fixed N , conditioned on Φ , the individual link capacities are i.i.d. Gaussian with the conditional mean and variance given in (7). By Lemma 1, through exploiting multiuser diversity, the overall system throughput is given by the solution of the following equation:

$$E_\Phi \left\{ Q \left(\frac{b_K - N(\log \rho + \log \Phi)}{\log e \sqrt{N/M}} \right) \right\} = \frac{1}{K}. \quad (19)$$

When Φ is log-normal distributed, (19) can be further simplified as (when $K \rightarrow \infty$)

$$e^{-f^2(b_K)/2} = \frac{1}{K}, \quad (20)$$

where

$$f(b_K) = \frac{b_K - N(\log \rho + \lambda \mu_L \log e)}{\log e \sqrt{\frac{N}{M} + N^2 \lambda^2 \sigma_L^2}}, \quad (21)$$

which leads to the following result.

Proposition 3: For large M , ρ and fixed N , and log-normal shadowing, as $K \rightarrow \infty$, the system capacity with optimal user scheduling converges in probability to

$$N(\log \rho + \lambda \mu_L \log e) + \log e \sqrt{2 \left(\frac{N}{M} + N^2 \lambda^2 \sigma_L^2 \right) \ln K}. \quad (22)$$

Remark: For pure Rayleigh fading, the corresponding result is given by [5]

$$N \log \rho + \log e \sqrt{\frac{2N}{M} \ln K}. \quad (23)$$

Note that (22) and (23) comprise two parts. The first part is related to the mean of individual link capacity, while the second one is related to its variance, which can also be viewed as the relative scheduling gain with respect to the round robin approach. As $M \rightarrow \infty$, it is observed that the relative scheduling gain is vanishing with pure Rayleigh fading (indicating a tradeoff between multiple antennas and multiuser diversity), which nonetheless remains significant when shadow fading is considered. This observation was originally made in [5] and is theoretically verified here.

Similar results can be obtained when both M and N are large while their ratio keeps fixed.

Proposition 4: For large M , N , ρ , with $M/N \rightarrow \beta$, and log-normal shadowing, as $K \rightarrow \infty$, the system capacity with optimal user scheduling converges in probability to

$$\left\{ \begin{aligned} & M \left[\left(\log \frac{\rho}{e} + \log \frac{1-\beta}{\beta} + \frac{1}{\beta} \log \frac{1}{1-\beta} \right) + \lambda \mu_L \log e \right] \\ & + \log e \sqrt{2 \left(\ln \frac{1}{1-\beta} + M^2 \lambda^2 \sigma_L^2 \right) \ln K}, \quad \beta < 1 \\ & N \left[\log \frac{\rho}{e} + \lambda \mu_L \log e \right] \\ & + \log e \sqrt{2 \left((\ln N + \gamma + 1) + N^2 \lambda^2 \sigma_L^2 \right) \ln K}, \quad \beta = 1 \\ & N \left[\left(\log \frac{\rho}{e} + (\beta - 1) \log \frac{\beta}{\beta - 1} \right) + \lambda \mu_L \log e \right] \\ & + \log e \sqrt{2 \left(\ln \frac{\beta}{\beta - 1} + N^2 \lambda^2 \sigma_L^2 \right) \ln K}, \quad \beta > 1. \end{aligned} \right. \quad (24)$$

Remark: It is interesting to observe the symmetry between results for outage capacity (Proposition 1 and 2) and multiuser scheduling gain (Proposition 3 and 4), which clearly reveals the contradicting roles of shadow fading.

The above discussion on multiuser scheduling gain refers to choosing the user with the best instantaneous channel capacity. This could be achieved through feedback from each user after required measurement and calculation. In practice, an appealing alternative is to choose the user only based on its local mean SNR Φ_k , which is much more energy- and bandwidth-efficient. Arbitrary mobility [13] or other methods in [4][12] can be introduced when fairness is a concern. The following result indicates that such a simple scheme incurs almost no loss in optimality.

Proposition 5: For large M , ρ and fixed N , and log-normal shadowing, as $K \rightarrow \infty$, the system capacity resulted from selecting the user with the largest Φ_k converges almost surely to

$$N(\log \rho + \lambda \mu_L \log e) + N \lambda \sigma_L \log e \sqrt{2 \ln K}. \quad (25)$$

Proof: As $M \rightarrow \infty$, the resultant system capacity converges almost surely to

$$N \log(1 + \rho \max \Phi_k) \approx N \log \rho + N \log \max \Phi_k \quad (26)$$

due to strong law of large numbers. Furthermore, it is known [11] that if Z_1, \dots, Z_K are i.i.d. Gaussian with mean μ and variance σ^2 , then as $K \rightarrow \infty$

$$\max_{1 \leq k \leq K} Z_k \xrightarrow{a.s.} \mu + \sigma \sqrt{2 \ln K}. \quad (27)$$

Therefore, for an i.i.d. log-normal sequence Φ_1, \dots, Φ_K , we have as $K \rightarrow \infty$

$$\max_{1 \leq k \leq K} \Phi_k \xrightarrow{a.s.} e^{\lambda \mu_L + \lambda \sigma_L \sqrt{2 \ln K}}. \quad (28)$$

Plugging (28) into (25) we got the desired results. ■

Remark: Comparison of Proposition 3 and 5 reveals that this simple selection rule is asymptotically optimal when $M \rightarrow \infty$. Intuitively with a large number of transmit antennas the small scale fading for each link is averaged out, and only shadow fading contributes to the dynamism of individual link quantities and resultant scheduling gain. Similar results can be obtained for the case when both M and N are large while their ratio keeps fixed, and are omitted here.

5. CONCLUSIONS

It is realized that there is a tradeoff between outage performance and scheduling gain for multiuser communications: the former rejects variation while the latter prefers. Shadow fading, largely ignored in MIMO

study, generally increases channel variation and changes the relevant results originally obtained only considering small-scale fading. The study of new arrangement of multiple antennas in MIMO systems, e.g., distributed MIMO systems [14][15], and its interaction with shadow fading, constitute our ongoing work.

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