ASYMPTOTIC ANALYSIS IN MIMO DIVERSITY SYSTEMS

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ABSTRACT

In this paper, through asymptotic analysis of the distribution of the largest singular value of a multi-input multi-output (MIMO) system, we obtain some succinct results for average error performance and capacity scaling law of MIMO maximum-ratio-transmission/maximum-ratio-combining (MRT/MRC) systems. Similar results are also obtained for two other MIMO diversity schemes, space-time block coding (STBC) and selection combining (SC). Our results reveal a simple connection with system parameters, providing good insights for the design of MIMO diversity systems.

1. INTRODUCTION

Multi-input multi-output (MIMO) systems can be exploited for spatial multiplexing or diversity gains. For diversity usage, joint maximum ratio transmission (MRT) and maximum ratio combining (MRC) provides the optimal performance reference [2]-[5]. With some restrictive assumptions on the beamforming vectors, the average output signal-to-noise ratio (SNR) of a MRT/MRC system is upper and lower bounded in [2], based on which the average symbol error rate (SER) and diversity order for BPSK modulation are approximately derived. With the restraining assumptions in [2] removed, it is well known that (for white Gaussian noise) the optimal transmit and receive beamformer are given by the principal right and left singular vector of the channel matrix $\mathbf{H}$, respectively; and the MIMO channel is transformed into a single-input single-output link with equivalent channel gain $\sigma_{\text{max}}$, the largest singular value of $\mathbf{H}$. For Rayleigh fading channels, the distribution of $\sigma_{\text{max}}$, already derived in [1], is revisited in [3] and expressed in an alternative form – a linear combination of Gamma functions. Based on this expression, the exact system SER is derived for general modulation schemes in [3], which nonetheless still involves complex numerical calculations. The distribution of $\sigma_{\text{max}}$ for Ricean fading is obtained in [4]. Unfortunately, results in [3] and [4] don’t easily lead one to an insightful understanding of the impact of the system parameters, including the number of transmit and receive antennas $M$ and $N$, on performance. For example, it is not readily clear that the diversity order of a MIMO MRT/MRC system is $MN$, though implied in both papers. Similarly, both papers show through simulations that when $M + N$ is fixed, a distribution of antennas between the transmitter and receiver with minimum $|M - N|$ gives the best performance, but don’t provide a rigorous justification.

In this paper, we obtain a succinct and accurate approximation for average SER of MIMO MRT/MRC systems at high SNR. Furthermore, we obtain a capacity scaling law for MIMO MRT/MRC systems when multiuser diversity is exploited [7]. A common theme in our study is the investigation of the approximate behavior of the distribution of $\sigma_{\text{max}}$ at the extremes, with the former at the origin and the latter at the tail. We also compare the MIMO MRT/MRC system with two other widely deployed MIMO diversity schemes: one is joint space-time block coding and maximum ratio combining (STBC/MRC) $^1$, and the other is selection combining at both ends (SC/SC). Our results reveal a simple connection with system parameters, providing good insights for the design of MIMO diversity systems.

2. SYSTEM MODEL

We assume a narrowband MIMO diversity system with $M$ transmit antennas and $N$ receive antennas:

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n} = \mathbf{Hw} \mu + \mathbf{n},$$

where $\mathbf{w} \in \mathbb{C}^{M \times 1}$ is the unit-norm transmit weight vector and $\mu$ is the transmitted symbol with power $P_t$, $\mathbf{y} \in \mathbb{C}^{N \times 1}$ is the received vector, $\mathbf{H} \in \mathbb{C}^{N \times M}$ is the channel matrix, and $\mathbf{n} \in \mathbb{C}^{N \times 1}$ is a zero-mean circularly symmetric complex Gaussian noise vector with variance $\sigma_n^2 / 2$ per real

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$^1$ In this paper, a full-rate STBC is assumed for simplicity.
dimension. We define \( \gamma = P_t / \sigma_n^2 \) as the average transmit SNR. For illustration purpose, independent and identically distributed Rayleigh fading is considered for \( \mathbf{H} \), but our analysis can be readily extended to the Ricean fading and correlated fading scenarios with appropriate distributions incorporated. When multiple MIMO users are involved, their channels are assumed independent with each other.

At the receiver side a weight vector \( \mathbf{w} \in \mathbb{C}^{N \times 1} \) is applied on \( \mathbf{y} \) to obtain a decision statistic for \( u \). For a MIMO MRT/MRC system, \( \mathbf{w} \) and \( \mathbf{w}_r \) are chosen to be the principal right and left singular vector corresponding to the largest singular value \( \sigma_{\text{max}} \) of \( \mathbf{H} \) to maximize the output SNR. For a STBC/MRC system, the transmitter doesn’t need channel state information (CSI) and the transmit power is equally allocated among the transmit antennas. As to a SC/SC system, one transmit antenna and one receive antenna are selected so that the resultant channel gain is maximized. This scheme requires less feedback than the MIMO MRT/MRC.

The cumulative distribution function (CDF) of \( \beta = \sigma_{\text{max}}^2 \) is given by [1]

\[
F(\beta) = \frac{\left| \Psi_s(\beta) \right|}{\prod_{t=0}^{s-1} \Gamma(t-k+1) \Gamma(s-k+1)}, \quad \beta \in (0, +\infty),
\]  
(2)

where \( s = \min(M, N) \), \( t = \max(M, N) \), and \( \Psi_s(\beta) \) is an \( s \times s \) Hankel matrix function with the \((i, j)\) entry given by \( \left( \Psi_s(\beta) \right)_{i,j} = \gamma(t-s+i+j-1, \beta) \), for \( i, j = 1, 2, \ldots, s \). Here \( \gamma(a, \beta) \) is the incomplete Gamma function defined as \( \gamma(a, \beta) = \int_0^\beta e^{-t} t^{a-1} dt \), and \( \Gamma(a) \) is the Gamma function defined as \( \Gamma(a) = \gamma(a, +\infty) \). The probability distribution function (PDF) of \( \beta \) can be derived as

\[
f(\beta) = F(\beta) \Gamma(\Psi_s(\beta) \Phi_s(\beta)), \quad \beta \in (0, +\infty),
\]  
(3)

where \( \Phi_s(\beta) \) is an \( s \times s \) matrix whose \((i,j)\)th entry is given by \( \left( \Phi_s(\beta) \right)_{i,j} = \beta^{i+j-1} e^{-\beta} \).

**3. SER APPROXIMATION AT HIGH SNR**

At high SNR, the conditional SER for lattice-based modulations can be represented by the Gaussian tail Q-function as \( P_e(\mathbf{H}) = M_o Q(\sqrt{\gamma N} \beta) \), where \( M_o \) is the number of the nearest neighboring constellation points, and \( \gamma \) is a positive fixed constant determined by the modulation and coding schemes [5]. To find a good approximation for the average SER \( P_e = E \{ P_e(\mathbf{H}) \} \) at high SNR, we need the following result for the behavior of \( f(\beta) \) at the origin.

**Lemma 1:**

\[
f(\beta) = \frac{MN!}{\prod_{k=0}^{t-1} k!} \beta^{MN-1} + o(\beta^{MN-1}), \quad \beta \to 0.
\]  
(4)

**Sketch of Proof:** By Maclaurin Series expansion

\[
\left( \Psi_s(\beta) \right)_{1,j} = \frac{1}{i-s+i+j-1} \beta^{i+j-1} + o(\beta^{i+j-1}),
\]  
(5)

we can obtain the approximation of \( \left| \Psi_s(\beta) \right| \) at \( \beta = 0 \) after some manipulation as

\[
\left| \Psi_s(\beta) \right| = \Phi_s^{MN} + o(\beta^{MN}),
\]  
(6)

with \( \Phi_{1,j} = 1/(t-s+i+j-1) \), for \( i, j = 1, 2, \ldots, s \). The determinant of \( \Phi \) can be obtained in a similar fashion as for a Hilbert matrix. After some algebra we get

\[
\Phi_s = \frac{\prod_{k=1}^{s-1} (k!)^2 ((\lambda+k)!)^2}{\prod_{k=0}^{s-1} (\lambda+k)!},
\]  
(7)

and

\[
F(\beta) = \frac{\prod_{k=0}^{t-1} k!}{\prod_{k=0}^{t-1} (t+k)!} \beta^{MN} + o(\beta^{MN}).
\]  
(8)

With Lemma 1, we establish the following result for high SNR SER approximation for MIMO MRT/MRC systems following Proposition I in [6].

**Proposition 1:** For MIMO MRT/MRC systems, the average SER at high SNR is given by

\[
P_e = \frac{2^s M_o^{\alpha_{\text{MRT/MRC}}} \Gamma(q^{\alpha_{\text{MRT/MRC}}}/2)}{\sqrt{\pi} \Gamma(\alpha_{\text{MRT/MRC}}) + 1} (\kappa \gamma)^{\alpha_{\text{MRT/MRC}}}(q^{\alpha_{\text{MRT/MRC}}}/2),
\]  
(9)

where

\[
\alpha_{\text{MRT/MRC}} = \frac{MN!}{\prod_{k=0}^{t-1} k!} \quad \text{and} \quad q^{\alpha_{\text{MRT/MRC}}} = MN - 1.
\]  
(10)

![Fig. 1 Approximate and simulated results for BPSK under different antenna configurations](image-url)
The validity of (9) is demonstrated in Fig. 1. Based on (9), one readily concludes that the optimal diversity order for MIMO diversity systems is $MN$. Therefore, if we keep $M + N$ fixed (a measure of system cost), even distribution (more precisely a smallest $|M - N|$) maximizes $MN$ and minimizes the system SER. On the other hand, when comparing two MIMO diversity systems with the same diversity order $MN$, the one with smaller $\alpha^{(MRT/MRC)}$ yields larger coding gain and thus smaller SER. We can conclude that in this scenario, $M + N$ should be made as large as possible, with the optimum achieved at $s = 1$ and $t = MN$. This conclusion is based on the following result regarding $\alpha^{(MRT/MRC)}$ as a function of $M$ and $N$ (or equivalently of $s$ and $t$), which we state without proof due to space limitations.

Lemma 2: Given four positive integers $s_1, t_1, s_2, t_2$, assume $s_1 \neq t_1 = s_2 \neq t_2$, and $s_1 < s_2 < t_2 < t_1$, then $\alpha^{(MRT/MRC)}(s_1, t_1) < \alpha^{(MRT/MRC)}(s_2, t_2)$.

Following a similar approach, we can obtain the corresponding parameters for the coding gain and diversity order for MIMO STBC/MRC and SC/SC systems (whose SERs assume same forms as (9)) as

\begin{equation}
\alpha^{(STBC/MRC)} = \frac{M^N}{(N-1)!}, \quad q^{(STBC/MRC)} = MN - 1, \quad (11)
\end{equation}

and

\begin{equation}
\alpha^{(SC/SC)} = MN, \quad q^{(SC/SC)} = MN - 1. \quad (12)
\end{equation}

Comparing (10), (11) and (12), we can see that all these MIMO diversity schemes achieve the same diversity order. Nonetheless, their error performances could still be dramatically different owing to different coding gains, as exhibited in Fig. 2. For example, when $M = 6$ and $N = 1$, our formulas predict a SNR gap of 4.7 dB between MRT/MRC ($\alpha^{(MRT/MRC)} = 1/120$) and SC/SC ($\alpha^{(SC/SC)} = 6$), and 7.8 dB between MRT/MRC and STBC/MRC ($\alpha^{(STBC/MRC)} = 388.8$) for uncoded BPSK systems (at high SNR), which agree well with simulation results (not shown). It is also observed that for the same diversity order, the performance of STBC worsens with the increase of transmit antennas.

4. CAPACITY SCALING LAW IN MULTIUSER SCENARIOS

It is well known that in a multiuser network, for delay-tolerant applications, system throughput can be improved through exploiting the multiuser diversity, i.e., selecting the user with the best channel at each time to communicate with [7]. In this section, we explore how the system capacity of a multiuser MIMO MRT/MRC system scales with the number of users $K$. To this end, the tail behavior of $f(\beta)$ is required, which we state below.

\begin{equation}
f(\beta) = e^{-\beta} B^{M_s-N-2}[a_i + O(1/\beta)], \quad \text{as } \beta \to \infty,
\end{equation}

where $a_i = \frac{1}{(M-1)!(N-1)!}$. 

Sketch of Proof: Use (3) and the fact that

\begin{equation}
\lim_{\beta \to \infty} \Psi_\alpha(\beta) = (t - s + i - j - 2)!.
\end{equation}

With Lemma 3, we derive in the following the capacity scaling law for multiuser MIMO MRT/MRC systems.

Proposition 2: When multiuser diversity is exploited in a $K$-user MIMO MRT/MRC system, the system capacity $C^{(MRT/MRC)}_K$ satisfies

\begin{equation}
\Pr \left[ C^{(MRT/MRC)}_K \leq C^{(MRT/MRC)}_M \leq C^{(MRT/MRC)}_U \right] \geq 1 - O(1/\log K), \quad \text{as } K \to \infty, \quad (14)
\end{equation}

where

\begin{equation}
C^{(MRT/MRC)}_K = \log_2 \left[ 1 + \gamma \left( \log a_K + (M + N - 1) \log \log K \right) \right] + O(\log \log K),
\end{equation}

\begin{equation}
C^{(MRT/MRC)}_U = \log_2 \left[ 1 + \gamma \left( \log a_K + (M + N - 1) \log \log K \right) \right] + O(\log \log K).
\end{equation}

Sketch of Proof: Define the growth function $g(\beta) = (1 - F(\beta))/f(\beta)$, with Lemma 3 we have

\begin{equation}
\lim_{\beta \to \infty} \frac{g(\beta)}{f(\beta)} = 1.
\end{equation}

Clearly $F(\beta)$ in (2) is less than 1 for all finite $\beta$ and is twice differentiable for all $\beta$. By (19) of [8]

\begin{equation}
\log \left[ \log F^{K} (l_k + xg(l_k)) \right] = -x + \frac{x^2}{2!} g(l_k) + \frac{x^3}{3!} \left[ g(l_k) g^{(2)}(l_k) - 2g^2(l_k) \right] + \ldots + \frac{e^{-x}}{2K} + \frac{5e^{-x} \ldots}{24K^2} + \ldots - \frac{1}{8K^3} e^{-x} + \ldots,
\end{equation}

Fig. 2 Coding gain parameter $\alpha$ for the same diversity order

\begin{equation}
\log [\log F^{K} (l_k + xg(l_k))] = -x + \frac{x^2}{2!} g(l_k) + \frac{x^3}{3!} \left[ g(l_k) g^{(2)}(l_k) - 2g^2(l_k) \right] + \ldots + \frac{e^{-x} \ldots}{2K} + \frac{5e^{-x} \ldots}{24K^2} + \ldots - \frac{1}{8K^3} e^{-x} + \ldots,
\end{equation}

(17)
where \( l_k \) is given by \( F(l_k) = 1 - 1/K \). Solving for \( l_k \) we can get
\[
l_k = \log a_K + (M + N - 2) \log \log a_K + O(\log \log \log K) = O(\log K).
\]

(18)

A close examination of \( g'(\beta) \) using Lemma 3 reveals
\[
g'(\beta) = O(1/\beta), \text{ and } \lim_{K \to \infty} [Kg'(l_k)] = +\infty.
\]

(19)

Therefore, the terms in the third line of (17) starting with the term \( e^{-x^2}/2K \) can be ignored [8]. Further exploiting (16), (18) and (19) in the second line of (17) with \( x = \log \log K \) yields
\[
\Pr \left\{ -\log \log K \leq \left( \max_{1 \leq k \leq K} \beta_k \right) - l_k \leq \log \log K \right\} \geq 1 - O \left( \frac{1}{\log K} \right).
\]

(20)

where \( \beta_k \) is the metric (the largest singular value squared) for the \( k \)th user. Proposition 2 follows with
\[
C_{k}^{(\text{MRT/MRC})} = \log_2 (1 + \max_{1 \leq k \leq K} \beta_k).
\]

We can obtain similar scaling laws for MIMO STBC/MRC and SC/SC systems as (14), which are dictated by
\[
C_{kL}^{(\text{STBC/MRC})} = \log_2 \left[ 1 + \gamma_1 \left( \frac{1}{M} \log b_K + (N - 2) \log \log K \right) + O(\log \log \log K) \right],
\]
\[
C_{kU}^{(\text{STBC/MRC})} = \log_2 \left[ 1 + \gamma_1 \left( \frac{1}{M} \log b_K + N \log \log K \right) + O(\log \log \log K) \right],
\]

(21)

where \( b_1 = \frac{M^{MN-1}}{(MN-1)!} \).

and
\[
C_{kL}^{(\text{SC/SC})} = \log_2 \left[ 1 + \gamma_1 \left( \log MNK - \log \log K \right) + O(\log \log \log K) \right],
\]
\[
C_{kU}^{(\text{SC/SC})} = \log_2 \left[ 1 + \gamma_1 \left( \log MNK + \log \log K \right) + O(\log \log \log K) \right].
\]

(22)

Some interesting observations are readily in order. From (21), a tradeoff between transmit diversity and multiuser diversity is seen, which has been observed by other researchers (e.g., [7][10]). Here we give a more rigorous proof in terms of convergence and reveal more details of the ultimate capacity relating to \( M \) and \( N \). It is also observed that the detrimental effect of multiple transmit antennas can be avoided by providing some feedback of CSI, as seen in the other two cases. In the optimal case ((15)), spatial diversity even contributes to the capacity of a MIMO system exploiting multiuser diversity, though at the second order.

5. CONCLUSIONS

In this paper, through asymptotic analysis of the distribution of the largest singular value of a MIMO system, we obtain some succinct results for average error performance of MIMO MRT/MRC systems in the single-user scenario, and capacity scaling law in the multiuser scenario when multiuser diversity is exploited. Our results provide a performance reference for MIMO diversity systems, facilitating various tradeoff studies in terms of system parameters and designs.

REFERENCES


2 It can be shown that (20) still holds for a more general condition \( g(x) = O(1/x^\delta) \) with \( \delta > 0 \).