On the Duality between Outage Capacity and Multiuser Scheduling Gain for MIMO Systems and the Impact of Shadow Fading*

Huaiyu Dai
Department of Electrical and Computer Engineering
NC State University
Raleigh, NC 27695
Email: Huaiyu_Dai@ncsu.edu

Abstract

Shadow fading is generally not explicitly studied for MIMO systems. In this paper, through asymptotic large-system analysis, the effect of shadow fading on individual link outage capacity and multiuser scheduling gain for MIMO systems are quantified, which admits a nice duality as in the scenario when only small-scale fading is considered. It is shown that shadow fading, while significantly limits the channel outage capacity as expected, actually enhances the multiuser scheduling gain, which is meaningful and obtainable for delay-tolerant applications.

* This research was supported in part by the National Science Foundation under Grant CCF-0515164

I. Introduction

Multipath fading is generally viewed as a detrimental effect for wireless communications, as it introduces dynamism on individual link quantities, which leads to outage and is especially unfavorable to real-time applications. Recently it is realized that fading can be beneficial as well due to induced independence and diversity. Indeed, it is the independence of parallel spatial subchannels created by rich scattering in the propagation environment that leads to the tremendous capacity growth in multi-input multi-output (MIMO) systems [1][2]. Another example is multiuser diversity, where independent fading across users actually contributes to the scheduling gain in a multiuser environment for delay-tolerant applications [3][4]. For MIMO systems considering only small-scale fading, it is shown in [5] that individual MIMO channels exhibit a "channel hardening" effect as the number of antennas grows, i.e., the variance of its instantaneous capacity decreases rapidly relative to the mean. Furthermore, outage capacity and multiuser scheduling gain actually admit a nice duality, with the former decreasing and the latter increasing linearly with the link standard deviation, thanks to the asymptotic Gaussianity of the instantaneous channel capacity in this scenario.

Current study of MIMO systems seldom explicitly addresses the shadow fading issues, though it is natural to expect severely diminished link quality when unfavorable shadowing is experienced. In [5], it is observed through simulations that shadow fading can nonetheless contribute a non-trivial scheduling gain even when the number of antennas is large, in contrast to the pure Rayleigh fading scenario. Motivated by this work, this paper intends to quantify the effect of shadow fading on individual link outage capacity and multiuser scheduling gain for MIMO systems. Note that due to shadow fading, the instantaneous channel capacity is asymptotically best described with a distribution which is conditional Gaussian. Nevertheless, we endeavor to reveal a similar duality between outage capacity and multiuser scheduling gain in this scenario. When shadow fading is modeled with the log-normal distribution, closed-form expressions can be obtained which theoretically verify the observations in [5].

This paper is organized as follows. Section II presents the system model. In Section III, some results in [5] are reviewed and the duality between outage capacity and multiuser scheduling gain is pointed out. Our main results are presented in Section IV, addressing the impact of shadow fading on both individual link outage capacity and multiuser scheduling gain for MIMO systems. Some numerical results are given in Section V and Finally Section VI concludes the paper.

II. System Model

Consider a multiuser MIMO system, with M antennas at the base station and N antennas at each of the K users. In this paper we mainly study the downlink scenario, but all discussions can be extended to the uplink as well. Each link is modeled as (user index is omitted for simplicity)

$$\mathbf{y} = \Phi^{1/2}\mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y}$ is the received vector, $\mathbf{x}$ is the transmitted vector with the total transmitted power ρ equally divided among transmit antennas, $\mathbf{H}$ is the channel matrix representing small-scale fading while Φ captures the common large-scale fading effect, and $\mathbf{n}$ is the noise vector. The entries of $\mathbf{H}$ and $\mathbf{n}$ are independent and identically distributed (i.i.d.) complex Gaussian with zero mean and unit variance. Note that $\Phi = 1$ represents the common pure Rayleigh fading model. We assume a block fading scenario where $\mathbf{H}$ changes independently from one block to another, and Φ changes independently with $\mathbf{H}$ at a lower pace.

Without loss of generality, we exclude the path loss effect and assume i.i.d shadow fading for different users. A widely accepted model for the shadow fading coefficient is the log-normal distribution $\Phi = e^{Y}$, where $Y \sim \mathcal{N}(\lambda\mu_L, (\lambda\sigma_L)^2)$ is a Gaussian random variable, with μ_L (dB) the area mean, and σ_L (dB) the decibel spread, typically ranging between 6-12 depending on the severity of the shadow fading, and $\lambda = \ln 10/10$. The cumulative distribution function (CDF) of Φ is given by

$$F_{\Phi}(x) = 1 - Q\left(\frac{\ln x - \lambda\mu_L}{\lambda\sigma_L}\right), \tag{2}$$

where $Q(\cdot)$ is the standard Gaussian tail function. We also have

$$E\{\Phi\} = e^{\lambda\mu_L + \frac{\lambda^2\sigma_L^2}{2}},\ E\{\Phi^2\} = e^{2\lambda\mu_L + 2\lambda^2\sigma_L^2}. \tag{3}$$

Our study focuses on two asymptotic scenarios: (1) large M and fixed N and (2) large M and N with their ratio fixed. Besides analytical tractability through laws of large numbers and random matrix theory, the study of large system performance also has practical advantages: what is revealed in the asymptotic limit is fundamental in nature, which may be concealed in the finite case by random fluctuations and other transient properties of the matrix entries; moreover, the convergence to the asymptotic limit is typically rather fast as the system size grows [5]-[7]. We also focus our study on the high SNR regimes.

In the following, when convergence of a sequence of random variables is involved, shorthand notation "*D*" stands for in distribution, "*P*" for in probability, and "*a.s.*" for almost surely. "log" is used for logarithm with an arbitrary base, and "ln" for base *e*.

III.Pure Rayleigh Fading

In this section, some relevant results for the asymptotic Gaussianity of the instantaneous MIMO channel capacity $C = \log\det\left(\mathbf{I} + \frac{\rho}{M}\Phi\mathbf{H}\mathbf{H}^H\right)$ with pure Rayleigh fading ($\Phi = 1$) are briefly introduced. We first restate Theorem 2 and Corollary 1 (ii) in [5] below.

Theorem 1: For large M, ρ and fixed *N*, as $M \to \infty$

$$\frac{\left[C - \mu_1\right]}{\sigma_1} \xrightarrow{D} \mathcal{N}(0,1), \tag{4}$$

where $\mu_1 = N\log(1+\rho) \approx N\log\rho$ and $\sigma_1 = \sqrt{\frac{N}{M}\frac{\rho\log e}{(1+\rho)}} \approx \sqrt{\frac{N}{M}}\log e$.

Theorem 2: For large M, N, ρ, as $M, N \to \infty$ with $M/N \to \beta$,

$$\frac{\left[C - \mu_2\right]}{\sigma_2} \xrightarrow{D} \mathcal{N}(0,1), \tag{5}$$

where (γ is the Euler constant)

$$\mu_2 = \begin{cases} M\left[\log\frac{\rho}{e} + \log\frac{1-\beta}{\beta} + \frac{1}{\beta}\log\frac{1}{1-\beta}\right] & \beta < 1 \\ N\log\frac{\rho}{e} & \beta = 1 \\ N\left[\log\frac{\rho}{e} + (\beta-1)\log\frac{\beta}{\beta-1}\right] & \beta > 1 \end{cases} \text{ and } \sigma_2 = \begin{cases} \log e\sqrt{\log\frac{1}{1-\beta}} & \beta < 1 \\ \log e\sqrt{\ln N + \gamma + 1} & \beta = 1 \\ \log e\sqrt{\log\frac{\beta}{\beta-1}} & \beta > 1. \end{cases}$$

Remark: In the following, we simply use the notations μ and σ when the discussion applies to both scenarios. It is readily observed that when shadow fading is explicitly considered, the instantaneous channel capacity conditioned on Φ is Gaussian distributed with mean $\mu(\Phi) = \min(M,N)\log\Phi + \mu$ and standard deviation σ.

Outage capacity and multiuser scheduling gain admit elegant expressions thanks to Gaussianity. Outage capacity $C^{(p)}$ with respect to an outage probability $p = P(C < C^{(p)})$ is given by

$$C^{(p)} = \mu - \sigma Q^{-1}(p). \tag{6}$$

As for the scheduling that always selects the user with the best link quality to communicate with [3][4], the resultant system capacity for a *K*-user system C_K is given by

$\max(X_1, X_2, \cdots, X_K)$, where $X_1, X_2, \cdots, X_K$ are i.i.d. Gaussian with mean μ and standard deviation σ. It is known from [11] that as $K \to \infty$

$$\frac{C_K}{\mu + \sigma\sqrt{2\ln K}} \xrightarrow{a.s.} 1. \tag{7}$$

Note that (7) is a stronger result than what is given in [5]. Therefore when *K* is sufficiently large, the system capacity with optimal user scheduling is well approximated by $\mu + \sigma\sqrt{2\ln K}$. In other words, the scheduling gain relative to the round-robin approach (for which a throughput of μ is obtained) grows linearly with σ, while at a much lower rate with *K*. The duality between $C^{(p)}$ and C_K can be observed from (6) and (7) with respect to μ and σ for MIMO systems with pure Rayleigh fading. In the following, we will show that similar results hold when shadow fading is explicitly considered, which facilitate the evaluation of the impact of shadow fading on performance of MIMO systems.

IV. Impact of Shadow Fading

Our main results on the impact of shadow fading are given below.

Theorem 3: If $Z_1, \cdots, Z_K$ are i.i.d. and, conditioned on some random variable θ, are Gaussian distributed with the conditional mean $\mu(\theta)$ and the conditional variance σ^2, which does not depend on θ, $\max_{1\le k\le K} Z_k - b_K \xrightarrow{P} 0$ as $K \to \infty$, where b_K is the solution to

$$E_\theta\left\{Q\left(\frac{b_K - \mu(\theta)}{\sigma}\right)\right\} = \frac{1}{K}. \tag{8}$$

Proof: We define the CDF and PDF of $\{Z_k\}$ as

$$F_Z(x) = E_\theta\{F_G(x;\theta)\} \text{ and } f_Z(x) = E_\theta\{f_G(x;\theta)\}, \tag{9}$$

where $F_G(x;\theta)$ and $f_G(x;\theta)$ are CDF and PDF of a Gaussian random variable with mean $\mu(\theta)$ and variance σ^2. Then as $x \to \infty$, the following approximation becomes accurate:

$$1 - F_G(x;\theta) \approx \frac{\sigma^2}{x} f_G(x;\theta) \text{ and } -f_G^{'}(x;\theta) \approx \frac{x}{\sigma^2} f_G(x;\theta). \tag{10}$$

One can readily show that

$$\lim_{x\to\infty} \frac{d}{dx}\left[\frac{1 - F_Z(x)}{f_Z(x)}\right]$$
$$= -1 + \lim_{x\to\infty} \frac{-(1 - F_Z(x)) f_Z^{'}(x)}{f_Z^2(x)}$$

$$
\begin{aligned}
&= -1 + \lim_{x\to\infty} \frac{-(1 - E_\theta\{F_G(x;\theta)\}) E_\theta\{ f_G^{'}(x;\theta)\}}{\left[E_\theta\{ f_G(x;\theta)\}\right]^2} \\
&= -1 + \lim_{x\to\infty} \frac{E_\theta\left\{\frac{\sigma^2}{x} f_G(x;\theta)\right\} E_\theta\left\{\frac{x}{\sigma^2} f_G(x;\theta)\right\}}{\left[E_\theta\{ f_G(x;\theta)\}\right]^2} \\
&= 0,
\end{aligned}
\tag{11}
$$

where we have assumed that derivation and expectation can be exchanged under some mild conditions. By Theorem 10.5.2 in [10], the standardized extreme $\left(\max_{1\le k\le K} Z_k - b_K\right)/a_K$ has a limiting distribution $G_3(x) = \exp(-e^{-x})$, where

$$
b_K = F_Z^{-1}(1 - 1/K) \text{ and } a_K = (K f_Z(b_K))^{-1}. \tag{12}
$$

It can be further shown that

$$
\lim_{K\to\infty} \frac{1}{a_K} = \lim_{K\to\infty} K f_Z(b_K) = \lim_{K\to\infty} \frac{f_Z(b_K)}{1 - F_Z(b_K)} = \infty, \tag{13}
$$

so $\forall \varepsilon > 0$,

$$
\lim_{K\to\infty} P\left(\left|\max_{1\le k\le K} Z_k - b_K\right| > \varepsilon\right) = 1 - \lim_{K\to\infty} G_3(\varepsilon / a_K) = 0. \tag{14}
$$

■

Remark: When K is sufficiently large, the system capacity with optimal user scheduling is well approximated by b_K, the solution to (8). Meanwhile, the outage capacity $C^{(p)}$ in this scenario is given by the solution to

$$
E_\theta\left\{Q\left(\frac{\mu(\theta) - C^{(p)}}{\sigma}\right)\right\} = p. \tag{15}
$$

We again observe the duality between $C^{(p)}$ and C_K in this scenario. When the distribution of θ is known, more concrete results can be obtained. One of such examples relevant to our study is given below.

Corollary 1: With log-normal shadowing modeled as (2), the MIMO system capacity with optimal user scheduling admits

$$
\frac{C_K}{(\mu + \min(M,N)\lambda\mu_L \log e) + \sqrt{2(\sigma^2 + \min^2(M,N)\lambda^2\sigma_L^2 \log^2 e)\ln K}} \xrightarrow{a.s.} 1. \tag{16}
$$

Meanwhile, the outage capacity is given by

$$
C^{(p)} = (\mu + \min(M,N)\lambda\mu_L \log e) - \sqrt{\sigma^2 + \min^2(M,N)\lambda^2\sigma_L^2 \log^2 e}\, Q^{-1}(p). \tag{17}
$$

In the above, μ and σ are the corresponding mean and standard deviation in pure Rayleigh fading scenario given in Theorem 1 and 2.

Proof: Based on the discussions in Section III and Theorem 3, $C_K - b_K \xrightarrow{P} 0$ as $K \to \infty$, where b_K is the solution to

$$E_\Phi\left\{Q\left(\frac{b_K - \min(M,N)\log\Phi - \mu}{\sigma}\right)\right\} = \frac{1}{K}, \tag{18}$$

where $\ln\Phi \sim \mathcal{N}(\lambda\mu_L, (\lambda\sigma_L)^2)$. For a normalized Gaussian random variable X, we have

$$E\{Q(\eta + \gamma X)\} = Q\left(\frac{\eta}{\sqrt{1+\gamma^2}}\right). \tag{19}$$

Therefore (18) can be further simplified as (when $K \to \infty$)[1]

$$Q(f(b_K)) = O(e^{-f^2(b_K)/2}) = \frac{1}{K}, \tag{20}$$

where

$$f(b_K) = \frac{b_K - \mu - \min(M,N)\lambda\mu_L \log e}{\log e\sqrt{\sigma^2 + \min^2(M,N)\lambda^2\sigma_L^2}}, \tag{21}$$

which leads to $b_K = (\mu + \min(M,N)\lambda\mu_L \log e) + \sqrt{2(\sigma^2 + \min^2(M,N)\lambda^2\sigma_L^2 \log^2 e)\ln K} + o(1)$.

We are left to show that $P\left(\lim_{K\to\infty}\frac{C_K}{b_K} = 1\right) = 1$, which is verified by Theorem 4.4.4 in [11] and the fact that for arbitrary $k > 1$

$$\sum_{n=1}^{\infty}[1 - F(kb_n)] = \sum_{n=1}^{\infty} Q(f(kb_n)) < \sum_{n=1}^{\infty} e^{-f^2(kb_n)/2} = C_1\sum_{n=1}^{\infty} C_2(n)\frac{1}{n^{k^2}} < \infty, \tag{22}$$

where $C_1 = \exp[-(k-1)^2\mu_S^2)^2/2]$, $C_2(n) = \exp[-k(k-1)\mu_S\sqrt{2\ln n}] = o(1)$, with $\mu_S = \mu + \min(M,N)\lambda\mu_L \log e$.

As for the outage capacity, (17) is readily obtained through (19) and

$$E_\Phi\left\{Q\left(\frac{\min(M,N)\log\Phi + \mu - C^{(p)}}{\sigma}\right)\right\} = p. \tag{23}$$

■

With the log-normal model, the impact of shadowing is succinctly represented as an increase in mean and variance of instantaneous MIMO link capacity by $\min(M,N)\lambda\mu_L \log e$, and $\min^2(M,N)\lambda^2\sigma_L^2\log^2 e$, respectively. The latter will enhance the multiuser scheduling gain while deteriorate the outage capacity.

---

[1] $g(K) = O(f(K))$ denotes $\lim_{K\to\infty}\frac{g(K)}{f(K)} = k,\ 0 < |k| < \infty$; $g(K) = o(f(K))$ denotes $\lim_{K\to\infty}\frac{g(K)}{f(K)} = 0$.

The above discussion on multiuser scheduling gain refers to choosing the user with the best instantaneous channel capacity. This could be achieved through feedback from each user after required measurement and calculation. In practice, an appealing alternative is to choose the user only based on its local mean SNR Φ_k, which is much more energy- and bandwidth-efficient. Arbitrary mobility [13] or other methods in [4][12] can be introduced when fairness is a concern. The following result indicates that such a simple scheme incurs little loss in optimality.

Corollary 2: With log-normal shadowing modeled as (2), the MIMO system capacity resulted from selecting the user with the largest local mean SNR admits

$$\frac{C_K}{(\mu+\min(M,N)\lambda\mu_L\log e)+\min(M,N)\lambda\sigma_L\log e\sqrt{2\ln K}} \xrightarrow{a.s.} 1. \qquad (24)$$

Proof: By strong law of large numbers and random matrix theory [8], as the number of antennas grow, the conditional instantaneous channel capacity at high SNR converges almost surely to $\min(M,N)\log\max\Phi_k+\mu$. Further, for an i.i.d. log-normal sequence $\Phi_1,\cdots,\Phi_K$, by (7) we have as $K\to\infty$

$$\max_{1\le k\le K}\Phi_k \xrightarrow{a.s.} e^{\lambda\mu_L+\lambda\sigma_L\sqrt{2\ln K}}. \qquad (25)$$

■

V. Numerical Results

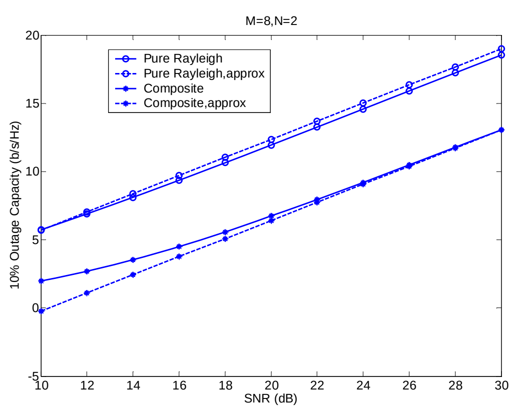


Fig. 1 Simulated and approximated Outage Capacity, large *M*

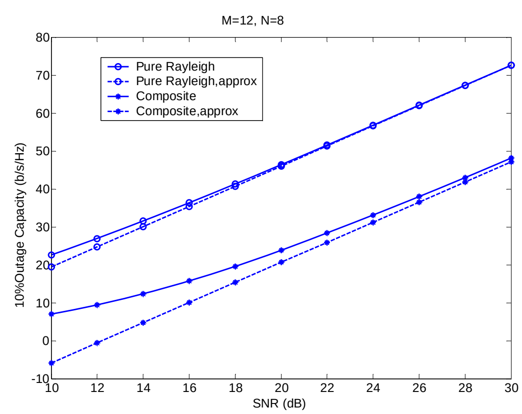
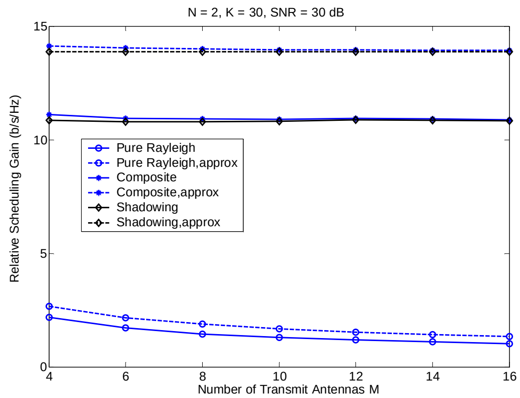


Fig. 2 Simulated and approximated Outage Capacity, large *M* and *N*

In this section, we provide some numerical examples to illustrate our results, for which we have assumed log-normal shadowing with $\mu_L=0$ and $\sigma_L=8$. From Fig. 1 and 2, we see that approximated results (6) and (17) match well with simulated ones at sufficiently high SNR, even for not so large number of antennas. Clearly, shadowing worsens outage capacity as expected, and we can accurately quantize this loss.



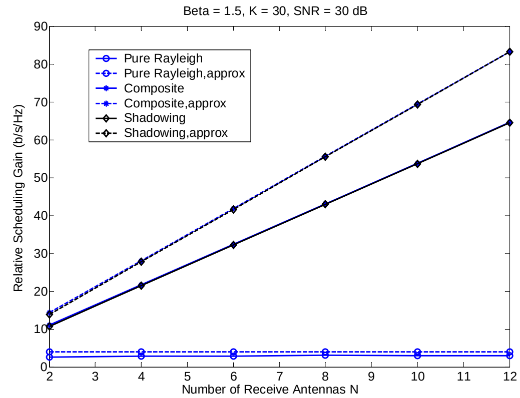


Fig. 3 Simulated and approximated scheduling gain, large *M*

Fig. 4 Simulated and approximated scheduling gain, large *M* and *N*

Results for multiuser scheduling gain (the second term in the denominator of (7), (16) and (24)) are shown in Fig. 3 and 4, which also match the simulated ones quite well. The discrepancy between simulated and approximated results when shadowing is considered is due to the relative small size of the user group. Note that the scheduling gain grows very slowly with *K*. In contrast, the link variance plays a more prominent role. As seen from Fig. 3, when *M* grows, the scheduling gain is vanishing with pure Rayleigh fading ($\sigma \approx \sqrt{\frac{N}{M}} \log e$, indicating a tradeoff between multiple antennas and multiuser diversity), which nonetheless remains significant when shadow fading is considered ($\sigma \approx N\lambda\sigma_L \log e$). Similarly, when both *M* and *N* are large, the scheduling gain is constant with pure Rayleigh fading ($\sigma \approx \log e \sqrt{\log \frac{\beta}{\beta - 1}}$ for $\beta > 1$), while grows with *N* with shadowing ($\sigma \approx N\lambda\sigma_L \log e$). These observations were originally made in [5] and are theoretically verified here. Finally, it is observed that the simple selection rule in Corollary 2 is asymptotically optimal. Intuitively with a large number of antennas the small scale fading for each link is nearly stabilized, and only shadow fading contributes to the dynamism of individual link quantities and resultant scheduling gain.

VI. Conclusions

It is realized that there is a tradeoff between outage performance and scheduling gain for multiuser communications: the former rejects variation while the latter prefers. Shadow fading, largely ignored in MIMO study, generally increases channel variation and changes the relevant results originally obtained only considering small-scale fading. The study of new arrangement of multiple antennas in MIMO systems, e.g., distributed MIMO systems [14][15], and its interaction with shadow fading, constitute our ongoing work.

References

[1] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels", *IEEE J. Select. Areas Commun.*, vol. 21, pp. 684-702, June 2003.

[2] D. Gesbert et al., "From theory to practice: An overview of MIMO space-time coded wireless systems", *IEEE J. Select. Areas Commun.*, vol. 21, pp. 281-302, Apr. 2003.

[3] R. Knopp and P. A. Humblet, "Information capacity and power control in single cell multiuser communications," *Proc. IEEE International Conference on Communications*, pp. 331-335, Seattle, WA, June 1995.

[4] P. Viswanath, D. N. C. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Transactions on Information Theory*, vol. 48, no. 6 , pp. 1277 -1294, June 2002.

[5] B. M. Hochwald, T. L. Marzetta and V. Tarokh, "Multi-Antenna channel hardening and its implications for rate feedback and scheduling", *IEEE Trans. Inform. Theory*, vol. 50, pp. 1893-1909, Sept. 2004.

[6] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585-595, Nov.-Dec. 1999.

[7] S. Shamai (Shitz) and S. Verdú, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp.1302-1327, May 2001.

[8] J. W. Silverstein and Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *J. Multivariate Anal.*, vol. 54, pp. 175-192, 1995.

[9] J. W. Silverstein and Z. D. Bai, "CLT of linear spectral statistics of large dimensional sample covariance matrices'", *Annals of Probability* 32(1A) (2004), pp. 553-605.

[10] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd edition, New York: John Wiley, 2003.

[11] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, 2nd edition, Malabar, Florida: Robert E. Krieger Publishing Company, 1987.

[12] M. Sharif and B. Hassibi, "Delay analysis of throughout optimal scheduling in broadcast fading channels," submitted to *IEEE Trans. Inform. Theory*.

[13] W. Li and H. Dai, "Optimal throughput and energy efficiency for wireless sensor networks: Multiple access and multi-packet reception," *European Journal on Wireless Communications and Networking*, 4th quarter 2005.

[14] W. Roh and A. Paulraj, "Outage performance of the distributed antenna systems in a composite fading channel," *Proc. 2002 Fall IEEE Conference on Vehicular Technology*, vol. 3, pp. 1520 -1524, Vancouver, Canada, Sept. 2002.

[15] H. Zhang and H. Dai, "On the Capacity of Distributed MIMO Systems," *Proc. 2004 Conference on Information Sciences and Systems* (CISS), Princeton University, Princeton, NJ, March 2004.