12.1 Complex Numbers

It was observed early in history that there are equations which are not satisfied by any real number. Examples are

$$x^2 = -3 \qquad \text{or} \qquad x^2 - 10x + 40 = 0.$$

This led to the invention of complex numbers.[1]

Definition

A **complex number** z is an ordered pair (x, y) of real numbers x, y and we write

$$z = (x, y).$$

We call x the **real part** of z and y the **imaginary part** of z and write

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501—1576), who found the formula for solving cubic equations. The term "complex number" was introduced by the great German mathematician CARL FRIEDRICH GAUSS (cf. the footnote in Sec. 4.4), who also paved the way for a general use of complex numbers.

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$$\text{Re } z = x, \qquad \text{Im } z = y.$$

Example: Re $(4, -3) = 4$ and Im $(4, -3) = -3$. Furthermore, we define two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ to be **equal** if and only if their real parts are equal and their imaginary parts are equal:

$$z_1 = z_2 \qquad \text{if and only if} \qquad x_1 = x_2 \text{ and } y_1 = y_2.$$

Addition of complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$(1) \qquad z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \; y_1 + y_2).$$

Multiplication is defined by

$$(2) \qquad z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \; x_1 y_2 + x_2 y_1).$$

We shall say more about these arithmetical operations and discuss examples below, but we first want to introduce a much more convenient form of writing complex numbers and then learn how to "see" these numbers by plotting them as points in the plane.

Representation in the Form $z = x + iy$

A complex number whose imaginary part is zero is of the form $(x, 0)$. For such numbers we simply have from (1) and (2)

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0),$$

as for real numbers. This suggests that we identify $(x, 0)$ with the real number x. Hence the complex number system is an *extension* of the real number system.

The complex number $(0, 1)$ is denoted by i,

$$i = (0, 1),$$

and is called the **imaginary unit.** We show that it has the property

$$(3) \qquad \boxed{i^2 = -1.}$$

Indeed, from (2) we have $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$.

Furthermore, for every real y we obtain from (2)

$$iy = (0, 1)(y, 0) = (0, y).$$

Combining this with the above $x = (x, 0)$ and using (1), that is,

$$(x, y) = (x, 0) + (0, y),$$

we see that we can write every complex number $z = (x, y)$ in the form

$$\boxed{z = x + iy}$$

or $z = x + yi$. This is done in practice almost exclusively.[2]

EXAMPLE 1. Complex numbers, their real and imaginary parts

$z = (4, -3) = 4 - 3i, \quad \text{Re}\,(4 - 3i) = 4, \quad \text{Im}\,(4 - 3i) = -3$

$z = (-\frac{1}{2}, 0) = -\frac{1}{2}, \quad \text{Re}\,(-\frac{1}{2}) = -\frac{1}{2}, \quad \text{Im}\,(-\frac{1}{2}) = 0$

$z = (0, \pi) = \pi i, \quad \text{Re}\,(\pi i) = 0, \quad \text{Im}\,(\pi i) = \pi$ ∎

Complex Plane

This is a geometrical representation of complex numbers as points in the plane. It is of great importance in applications. The idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis,** and the vertical y-axis, called the **imaginary axis.** On both axes we choose the same unit of length (Fig. 295). This is called a **Cartesian coordinate system.**[3] We now plot $z = (x, y) = x + iy$ as the point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane** or *Argand diagram.*[4] Figure 296 shows an example.

Instead of saying "the point represented by z in the complex plane" we say briefly and simply *"the point z in the complex plane."* This will cause no misunderstandings.

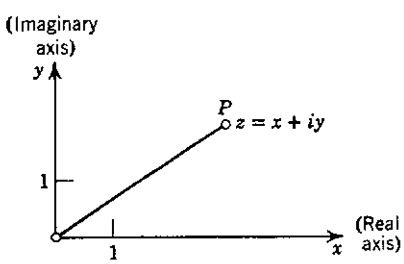

Fig. 295. The complex plane

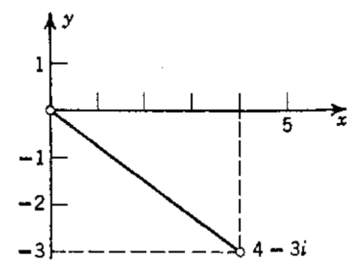

Fig. 296. The number $4 - 3i$ in the complex plane

[2]Electrical engineers often write j for i, to reserve the letter i for the electrical current.

[3]Named after the French philosopher and mathematician RENATUS CARTESIUS (latinized for RENÉ DESCARTES (1596—1650), who invented analytic geometry. His basic work *Géométrie* appeared in 1637, as an appendix to his *Discours de la méthode*.

[4]JEAN ROBERT ARGAND (1768—1822), French mathematician, was born in Geneva and later became a librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745—1818), who was surveyor of the Danish Academy of Sciences.

Arithmetic Operations

We can now make use of the notation $z = x + iy$ and of the complex plane.

Addition. The sum of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ can now be written

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{4}$$

Example: $(5 + i) + (1 + 3i) = 6 + 4i$. We see that addition of complex numbers is in accordance with the "*parallelogram law*" by which forces are added in mechanics (Fig. 297).

Subtraction is defined to be the inverse operation of addition. That is, the difference $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Obviously (cf. Fig. 298)

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{5}$$

Example: $(5 + i) - (1 + 3i) = 4 - 2i$.

Multiplication. The product $z_1 z_2$ in (2) can now be written

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \tag{6}$$

This is easy to remember since it is obtained formally by the rules of arithmetic for real numbers and using (3), that is, $i^2 = -1$.

Example: $(5 + i)(1 + 3i) = 5 + 15i + i + 3i^2 = 2 + 16i$.

Division is defined to be the inverse operation of multiplication. That is, the quotient $z = z_1/z_2$ is the complex number $z = x + iy$ for which

$$z_1 = zz_2 = (x + iy)(x_2 + iy_2) \qquad (z_2 \neq 0). \tag{7}$$

We show that for $z_2 \neq 0$ the quotient $z = x + iy = z_1/z_2$ is given by

$$x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0). \tag{8*}$$

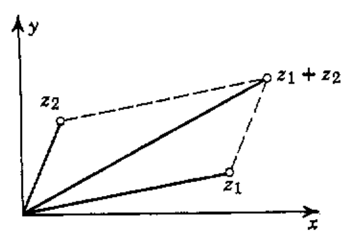

Fig. 297. Addition of complex numbers

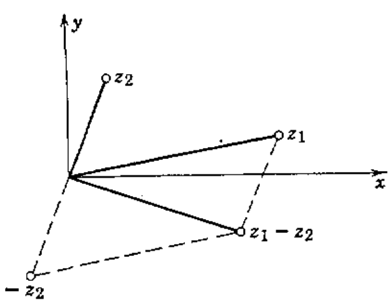

Fig. 298. Subtraction of complex numbers

The ***practical rule*** for getting (8*) is the multiplication of both the numerator and the denominator of the quotient z_1/z_2 by $x_2 - iy_2$ and simplification:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2} + i\frac{x_2y_1 - x_1y_2}{{x_2}^2 + {y_2}^2}. \tag{8}$$

Example: if $z_1 = 9 - 8i$ and $z_2 = 5 + 2i$, then

$$\frac{9 - 8i}{5 + 2i} = \frac{(9 - 8i)(5 - 2i)}{(5 + 2i)(5 - 2i)} = \frac{45 - 18i - 40i - 16}{25 + 4} = 1 - 2i.$$

The reader may check this result by showing that

$$zz_2 = (1 - 2i)(5 + 2i) = 9 - 8i = z_1.$$

A proof of (8*) runs as follows. From (6) we see that (7) can be written

$$x_1 + iy_1 = (x_2x - y_2y) + i(y_2x + x_2y).$$

By the definition of equality the real parts and the imaginary parts on both sides must be equal:

$$x_1 = x_2x - y_2y$$

$$y_1 = y_2x + x_2y.$$

This is a system of two linear equations in the unknowns x and y. Assuming that x_2 and y_2 are not both zero (briefly written $z_2 \neq 0$), we obtain the unique solution (8*). ▮

Properties of the Arithmetic Operations

From the familiar laws for *real* numbers we obtain for any complex numbers z_1, z_2, z_3, z the following laws (where $-z = -x - iy$):

$$\left.\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1z_2 &= z_2z_1 \end{aligned}\right\} \quad \text{(Commutative laws)}$$

$$\left.\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \\ (z_1z_2)z_3 &= z_1(z_2z_3) \end{aligned}\right\} \quad \text{(Associative laws)} \tag{9}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad \text{(Distributive law)}$$

$$0 + z = z + 0 = z$$

$$z + (-z) = (-z) + z = 0$$

$$z \cdot 1 = z$$

Complex Conjugate Numbers

Let $z = x + iy$ be any complex number. Then $x - iy$ is called the **conjugate** of z and is denoted by $\bar{z}$. Thus,

$$z = x + iy, \qquad \bar{z} = x - iy.$$

Example: the conjugate of $z = 5 + 2i$ is $\bar{z} = 5 - 2i$ (Fig. 299).

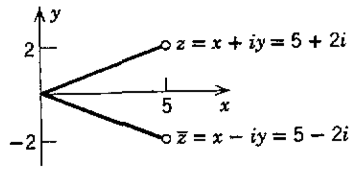

Fig. 299. Complex conjugate numbers

Conjugates are useful since $z\bar{z} = x^2 + y^2$ is real, a property we used in the above division. Moreover, addition and subtraction yields $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$, so that we can express the real part and the imaginary part of z by the important formulas

$$\text{(10)} \qquad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

Example: if $z = 6 - 5i$, then we have $\bar{z} = 6 + 5i$ and from (10) we obtain $x = \frac{1}{2}(6 - 5i + 6 + 5i) = 6$ and $y = -5$.

z is real if and only if $y = 0$, hence $\bar{z} = z$ by (10).

z is said to be **pure imaginary** if and only if $x = 0$, hence $\bar{z} = -z$. Then z corresponds to a point on the imaginary axis.

Working with conjugates is easy, since we have

$$\text{(11)} \qquad \begin{aligned} \overline{(z_1 + z_2)} &= \bar{z}_1 + \bar{z}_2, \qquad & \overline{(z_1 - z_2)} &= \bar{z}_1 - \bar{z}_2, \\ \overline{(z_1 z_2)} &= \bar{z}_1 \bar{z}_2, \qquad & \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

In this section we were mainly concerned with complex numbers, their arithmetic operations and their representation as points in the complex plane, a key idea for great progress in early complex analysis, conceptually and technically. In the complex plane we used rectangular xy-coordinates. In the next section we discuss the use of **polar coordinates** in the complex plane and situations in which polar coordinates are advantageous.

Problems for Sec. 12.1

1. **(Powers of the imaginary unit)** Show that

$$\text{(12)} \qquad \begin{gathered} i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \cdots \\ \frac{1}{i} = -i, \quad \frac{1}{i^2} = -1, \quad \frac{1}{i^3} = i, \cdots. \end{gathered}$$

Let $z_1 = 3 + 4i$ and $z_2 = 5 - 2i$. Find (in the form $x + iy$)

2. $(z_1 - z_2)^2$ **3.** z_1/z_2 **4.** $1/z_1^{\,2}$ **5.** $z_2/2z_1$

Find, in the form $x + iy$,

6. $(7 - 3i) - (-2 + 4i)$ **7.** $(2 + 4i)^2$

8. $(3 + 5i)(3 - 5i)$ **9.** $(5 + 2i)i$

10. $(1 - i)^4$ **11.** $\dfrac{11 + 2i}{4 + 3i}$ **12.** $\dfrac{52.5 - 12.5i}{3 - i}$ **13.** $\dfrac{101}{10 - i}$

Find:

14. $\operatorname{Re} \dfrac{1}{2 + i}$ **15.** $\operatorname{Im} \dfrac{2 + i}{3 + 4i}$ **16.** $\operatorname{Re} \dfrac{(1 + i)^2}{3 + 2i}$ **17.** $\operatorname{Im} \dfrac{2 - i}{4 - 3i}$

18. $\operatorname{Re} z^3$, $(\operatorname{Re} z)^3$ **19.** $\operatorname{Im} z^4$, $(\operatorname{Im} z^2)^2$ **20.** $\operatorname{Im} (1/z)$ **21.** $\operatorname{Re} (z/\bar{z})$

Prove:

22. The distributive law in (9).

23. z is pure imaginary if and only if $\bar{z} = -z$.

24. $\overline{iz} = -i\bar{z}$, $\operatorname{Re}(iz) = -\operatorname{Im} z$, $\operatorname{Im}(iz) = \operatorname{Re} z$.

25. If a product of two complex numbers is zero, at least one factor must be zero.

12.2 Polar Form of Complex Numbers. Powers and Roots

It is often practical to express complex numbers $z = x + iy$ in terms of polar coordinates r, θ. These are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

By substituting this we obtain the **polar form** of z,

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 300). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 301).

θ is called the **argument** of z and is denoted by $\arg z$. Thus (Fig. 300)

$$\theta = \arg z = \arctan \frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 300. Here, as in calculus, *all angles are measured in* ***radians*** *and positive in the counterclockwise sense.*

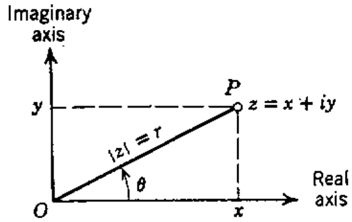

Fig. 300. Complex plane, polar form of a complex number

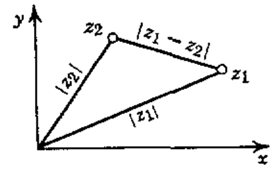

Fig. 301. Distance between two points in the complex plane

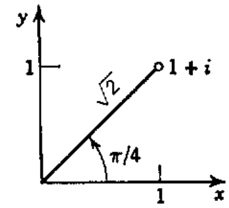

Fig. 302. Example 1

For $z = 0$ this angle θ is undefined. (Why?) For given $z \neq 0$ it is determined only up to integer multiples of 2π. The value of θ that lies in the interval $-\pi < \theta \leqq \pi$ is called the **principal value** of the argument of z ($\neq 0$) and is denoted by Arg z. Thus $\theta = \text{Arg } z$ satisfies by definition

$$-\pi < \text{Arg } z \leqq \pi.$$

EXAMPLE 1. Polar form of complex numbers. Principal value

Let $z = 1 + i$ (cf. Fig. 302). Then

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \qquad |z| = \sqrt{2}, \qquad \arg z = \frac{\pi}{4} \pm 2n\pi \qquad (n = 0, 1, \cdots).$$

The principal value of the argument is Arg $z = \pi/4$. Other values are $-7\pi/4$, $9\pi/4$, etc.

EXAMPLE 2. Polar form of complex numbers. Principal value

Let $z = 3 + 3\sqrt{3}\, i$. Then $z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, the absolute value of z is $|z| = 6$, and the principal value of arg z is Arg $z = \pi/3$. ∎

Caution! In using (4), we must pay attention to the quadrant in which z lies, since $\tan\theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan\theta_1 = \tan\theta_2 = 1$.

Triangle inequality

For any complex numbers we have the important **triangle inequality**

$$|z_1 + z_2| \leqq |z_1| + |z_2| \qquad \text{(Fig. 303)} \tag{5}$$

which we shall use quite frequently. This inequality follows by noting that

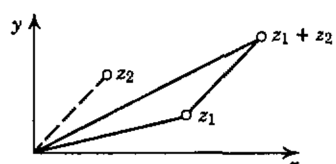

Fig. 303. Triangle inequality

the three points 0, z_1 and $z_1 + z_2$ are the vertices of a triangle[5] (Fig. 303) with sides $|z_1|$, $|z_2|$ and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 45).

Example: if $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$

By induction the triangle inequality can be extended to arbitrary sums:

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

Multiplication and Division in Polar Form

This will give us a better understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then, by (6), Sec. 12.1, the product is at first

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in Appendix 3.1] now yield

$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].} \tag{7}$$

Taking absolute values and arguments on both sides, we thus obtain the important rules

$$|z_1 z_2| = |z_1|\,|z_2| \tag{8}$$

and

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

We now turn to *division*. The quotient $z = z_1/z_2$ is the number z satisfying $zz_2 = z_1$. Hence $|zz_2| = |z|\,|z_2| = |z_1|$, $\arg(zz_2) = \arg z + \arg z_2 = \arg z_1$. This yields

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0) \tag{10}$$

and

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

[5]Which degenerates if z_1 and z_2 lie on the same straight line through the origin.

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$g\ z + \arg z_2 = \arg z_1.$

$(z_2 \neq 0)$

(up to multiples of 2π).

through the origin.

By combining these two formulas (10) and (11) we also have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \tag{12}$$

EXAMPLE 3. Illustration of formulas (8)-(11)

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = 2/3 + (2/3)i$. Hence

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1|\,|z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg} z_1 z_2 = -\frac{3\pi}{4} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi,$$

$$\operatorname{Arg}(z_1/z_2) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

Integer powers of z

From (7) and (12) we have

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta),$$

$$z^{-2} = r^{-2}[\cos(-2\theta) + i \sin(-2\theta)]$$

and, more generally, for any integer n,

$$\boxed{z^n = r^n(\cos n\theta + i \sin n\theta).} \tag{13}$$

EXAMPLE 4. Formula of De Moivre

For $|z| = r = 1$, formula (13) yields the so-called **formula of De Moivre**[6]

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta. \tag{13*}$$

This formula is useful for expressing $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$. For instance when $n = 2$ and we take the real and imaginary parts on both sides of (13*), we get the familiar formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\cos\theta \sin\theta.$$

This illustrates the general fact that *complex* methods often simplify the derivation of *real* formulas. ∎

Roots

If $z = w^n$ $(n = 1, 2, \cdots)$, then to each value of w there corresponds one value of z. We shall immediately see that to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an ***n*th root** of z, and we write

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is *multivalued*, namely, *n-valued*, in contrast to the usual

[6] ABRAHAM DE MOIVRE (1667—1754), French mathematician, who introduced imaginary quantities in trigonometry and contributed to probability theory (cf. Sec. 23.7).

conventions made in *real* calculus. The n values of $\sqrt[n]{z}$ can easily be determined as follows. In terms of polar forms for z and

$$w = R(\cos\phi + i\sin\phi),$$

the equation $w^n = z$ becomes

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta).$$

By equating the absolute values on both sides we have

$$R^n = r, \qquad \text{thus} \qquad R = \sqrt[n]{r}$$

where the root is real positive and thus uniquely determined. By equating the arguments we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n-1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$, hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\text{(15)} \qquad \boxed{\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)} \quad k = 0, 1, \cdots, n-1.$$

These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides.

The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** *of* $w = \sqrt[n]{z}$.

EXAMPLE 5. Square root

From (15) it follows that $w = \sqrt{z}$ has the two values

$$\text{(16a)} \qquad w_1 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and

$$\text{(16b)} \qquad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1$$

which lie symmetric with respect to the origin. For instance, the square root of $4i$ has the values

$$\sqrt{4i} = \pm 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm(\sqrt{2} + i\sqrt{2}).$$

From (16) we can obtain the much more practical formula

$$\text{(17)} \qquad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)i\sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with the positive sign. This follows from (16) if we use the trigonometric identities

$$\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 + \cos \theta)}, \qquad \sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 - \cos \theta)},$$

multiply them by $\sqrt{r}$,

$$\sqrt{r} \cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r + r \cos \theta)}, \qquad \sqrt{r} \sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r - r \cos \theta)},$$

use $r \cos \theta = x$, and finally choose the sign of Im $\sqrt{z}$ so that sign $[(\text{Re}\,\sqrt{z})(\text{Im}\,\sqrt{z})] = \text{sign}\, y$ (why?).

EXAMPLE 6. Complex quadratic equation
Solve $z^2 - (5 + i)z + 8 + i = 0$

Solution.

$$z = \tfrac{1}{2}(5 + i) \pm \sqrt{\tfrac{1}{4}(5 + i)^2 - 8 - i} = \tfrac{1}{2}(5 + i) \pm \sqrt{-2 + \tfrac{3}{2}i}$$

$$= \tfrac{1}{2}(5 + i) \pm [\sqrt{\tfrac{1}{2}(\tfrac{5}{2} + (-2))} + i\sqrt{\tfrac{1}{2}(\tfrac{5}{2} - (-2))}]$$

$$= \tfrac{1}{2}(5 + i) \pm [\tfrac{1}{2} + \tfrac{3}{2}i] = \begin{cases} 3 + 2i \\ 2 - i \end{cases}$$

EXAMPLE 7. Cube root of a positive real number
If z is *positive real*, then $w = \sqrt[3]{z}$ has the real value $\sqrt[3]{r}$ and the complex conjugate values

$$\sqrt[3]{r}\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) = \sqrt[3]{r}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

and

$$\sqrt[3]{r}\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) = \sqrt[3]{r}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right).$$

For instance, $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$ (Fig. 304). These are the roots of the equation $w^3 = 1$.

EXAMPLE 8. *n*th root of unity
Solve the equation $z^n = 1$.

Solution. From (15) we obtain

$$\sqrt[n]{1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1. \tag{18}$$

If ω denotes the value corresponding to $k = 1$, then the n values of $\sqrt[n]{1}$ can be written as $1, \omega, \omega^2, \cdots, \omega^{n-1}$. These values are the vertices of a regular polygon of n sides inscribed in the unit circle, with one vertex at the point 1. Each of these n values is called an **nth root of unity.** For instance, $\sqrt[4]{1}$ has the values $1, i, -1$, and $-i$ (Fig. 305). Figure 306 shows $\sqrt[5]{1}$.

If w_1 is any nth root of an *arbitrary* complex number z, then the n values of $\sqrt[n]{z}$ are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1}$$

since multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. ∎

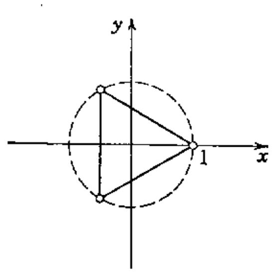

Fig. 304. $\sqrt[3]{1}$

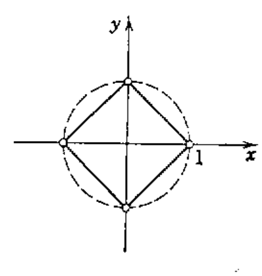

Fig. 305. $\sqrt[4]{1}$

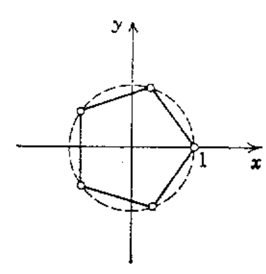

Fig. 306. $\sqrt[5]{1}$

The student should do the problems related to the polar representation with particular care, since we shall need this representation quite often in our work. In the next section we discuss some curves and regions in the complex plane which we shall also need in the chapters on complex analysis.

Problems for Sec. 12.2

1. **(Multiplication by i)** Show that multiplication of a complex number by i corresponds to a counterclockwise rotation of the corresponding vector through the angle $\pi/2$.

Find

2. $|1 - i|^2$ **3.** $|-\frac{3}{2}i|$ **4.** $|\cos\theta + i\sin\theta|$ **5.** $|6 - 8i|$

6. $\left|\dfrac{1 + 4i}{4 + i}\right|$ **7.** $\left|\dfrac{z}{\bar{z}}\right|$ **8.** $\left|\dfrac{(3 + 4i)^4}{(3 - 4i)^3}\right|$ **9.** $\left|\dfrac{1}{7 - \pi i}\right|$

Determine the principal value of the arguments of

10. $-2 + 2i$ **11.** -4 **12.** $-3i/2$ **13.** $1 - i\sqrt{3}$

Represent in polar form:

14. $1 + i$ **15.** $-4 + 4i$ **16.** $4i$ **17.** -7

18. $\dfrac{2 + 2i}{1 - i}$ **19.** $\dfrac{i\sqrt{2}}{3 + 3i}$ **20.** $\dfrac{6 + 8i}{4 - 3i}$ **21.** $\dfrac{3 + i\sqrt{2}}{-\sqrt{2} - 2i/3}$

Represent in the form $x + iy$:

22. $4\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$ **23.** $\sqrt{8}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

24. $\pi(\cos\pi + i\sin\pi)$ **25.** $\sqrt{50}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

Find all values of the following roots and plot them in the complex plane.

26. $\sqrt{i}$ **27.** $\sqrt{-i}$ **28.** $\sqrt{-4}$ **29.** $\sqrt{1 - i\sqrt{3}}$

30. $\sqrt[4]{-1}$ **31.** $\sqrt{3 + 4i}$ **32.** $\sqrt{-5 + 12i}$ **33.** $\sqrt{-8 - 6i}$

34. $\sqrt[3]{1 + i}$ **35.** $\sqrt[5]{-1}$ **36.** $\sqrt[6]{-1}$ **37.** $\sqrt[8]{1}$

Find and plot all solutions of the following equations.

38. $z^2 + z + 1 = i$ **39.** $z^2 - 3z + 3 = i$

40. $z^2 - (5 + i)z + 8 + i = 0$ **41.** $z^4 - 3(1 + 2i)z^2 = 8 - 6i$

42. **(Parallelogram equality)** Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. This is called the *parallelogram equality*. Why?

43. Prove the following useful inequalities, which we shall need from time to time.

(19) $$|\text{Re } z| \leqq |z|, \qquad |\text{Im } z| \leqq |z|$$

44. Let P be a regular polygon of n sides with vertices on the unit circle. Find the product of the lengths of the $n - 1$ straight-line segments that join a fixed vertex of P with the $n - 1$ other vertices.

45. Prove the triangle inequality.