

## 405 Special Matrices

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1. column vector  $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
2. row vector  $\mathbf{x}^T = [x_1, \dots, x_n]$
3. unit vectors  $\mathbf{e}_i = i \rightarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ,  
 $\mathbf{e}_i^T = [0, \dots, 0, 1, 0, \dots, 0]$ ,
4.  $\mathbf{e}^T = [1, 1, \dots, 1] = \mathbf{1}^T$
5. m by n zero matrix  $0_{mn}$
6.  $E_{ij} = \mathbf{e}_i \mathbf{e}_j^T = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \hline 0 & \dots & 1 & \dots & \\ \hline \vdots & & & & \\ 0 & \dots & 0 & & 0 \end{bmatrix}$
7.  $J = J_{m \times n} = \mathbf{e} \mathbf{e}^T = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & \\ \vdots & & & \\ 1 & \dots & & 1 \end{bmatrix}$
8. Identity  $I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \dots & \\ 0 & & & 1 \end{bmatrix}$
9. Scalar matrix  $cI$
10. Diagonal matrix  $D = \begin{bmatrix} d_1 & 0 & & 0 \\ & d_2 & & \\ & & \dots & \\ 0 & & & d_n \end{bmatrix}$   
 $= \text{diag}(d_1, \dots, d_n)$
11. Upper Triangular matrix  
 $T = \begin{bmatrix} t_{11} & t_{12} & & ? \\ & t_{22} & & \\ & & \dots & \\ 0 & & & t_{nn} \end{bmatrix}$   
 Lower Triangular matrix  $L = T^T$ .
12. Strictly Upper Triangular  
 $= \text{Upper matrix } U = \begin{bmatrix} 0 & u_{12} & & ? \\ 0 & 0 & u_{23} & \\ & & \dots & \\ 0 & & & 0 \end{bmatrix}$
13. Hessenberg matrix  
 $H = \begin{bmatrix} * & & & ? \\ * & * & \dots & ? \\ & \dots & \dots & \\ 0 & & * & * \end{bmatrix}$
14. Permutation matrix  $P = [\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}]$
15. Non-negative matrix  $A$ ,  $a_{ij} \geq 0$  (real)
16. Symmetric matrix  $A^T = A$
17. Skew-symmetric  $A^T = -A$
18. Hermitian  $A^* = A$
19. Skew-Hermitian  $A^* = -A$
20. Unitary  $A^* = A^{-1}$
21. Real Unitary = Real Orthogonal  
 $A^T = A^{-1} = \text{real}$
22. Normal matrix  $AA^* = A^*A$
23. Group matrix  $r(A) = r(A^2)$
24. rank-one matrix  $A = \mathbf{u} \mathbf{v}^T$
25. Idempotent matrix  $E^2 = E$
26. Projection = Hermitian idempotent  
 $P^* = P = P^2$
27. Involutory  $A^2 = I$
28. Nilpotent  $B^n = 0$
29. Stochastic matrix  $P \geq 0 \quad P \mathbf{e} = \mathbf{e}$

30. (i) A quadratic form:

$$q = \mathbf{x}^T A \mathbf{x} = \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} x_j$$

(ii) A Hermitian form:

$$h = \mathbf{x}^* A \mathbf{x} = \sum_{i=1}^n \sum_{j=1}^n \bar{x}_i a_{ij} x_j$$

(iii) Euclidean norm:

$$\|\mathbf{x}\|^2 = \mathbf{x}^* \mathbf{x} = |x_1|^2 + \dots + |x_n|^2$$

(iv) A bilinear form:

$$b = \mathbf{x}^T A \mathbf{y} = \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j$$

(v)  $\mathbf{e}_i^T A \mathbf{e}_j = a_{ij}$

31. Complex Positive Semi- Definite

$$\mathbf{x}^* A \mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{C}^n$$

32. Complex Positive Definite

$$\mathbf{x}^* A \mathbf{x} > 0, \forall \mathbf{x} \neq \mathbf{0}, \mathbf{x} \in \mathbb{C}^n$$

33. Real Positive Semi- Definite

$$\mathbf{x}^T A \mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$$

34. Real Positive Definite

$$\mathbf{x}^T A \mathbf{x} > 0, \forall \mathbf{x} \neq \mathbf{0}, \mathbf{x} \in \mathbb{R}^n$$

35. EP matrix  $R(A) = R(A^*)$

36. Reducible matrix A:

$$P^{-1} A P = \begin{bmatrix} A_1 & A_3 \\ 0 & A_4 \end{bmatrix} \text{ with } A_1, A_4 \text{ square} \\ \text{and } P \text{ a permutation matrix.}$$

37. Vandermonde Matrix (a,b,...,d distinct):

$$V = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & a & b & \dots & d \\ 1 & a^2 & b^2 & \dots & d^2 \\ \vdots & & & & \\ 1 & a^{n-1} & b^{n-1} & \dots & d^{n-1} \end{bmatrix}$$

38. Discrete Fourier Transform with

$$\omega = \exp(2\pi i/n)$$

$$W = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}$$