MA 405 Matrix Problems

collected by R. Hartwig Jan 25 2005

1 General Matrix Manipulation

1. (i) If $\beta A = 0$ show that either A = 0 or $\beta = 0$
   (ii) show that $A(\gamma B) = \gamma(AB)$, for all $\gamma, \beta \in \mathbb{F}$. What scalar property did you use ?

2. (a) Compute $(A+B)(A-B)$. When is equal to $A^2 - B^2$? Is the converse true ?
   (b) Compute $(A+B)^3$. What sizes must A and B have ?

3. Does $X^2 = I$ imply that X = I ? Find an X such that $X^2 = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}$

4. Show that the product of two diagonal matrices is again diagonal.
   What about two upper-triangular/strictly uppertriangular/Hessenberg matrices ?

5. If $\mathbf{x} = [1+i, 2-i]^T$, compute $\mathbf{x}^*\mathbf{x}$. If $\mathbf{y} = [a_1 + ib_1, ..., a_n + ib_n]^T$, with a_i, b_i real, what is $\mathbf{y}^*\mathbf{y}$?

6. Simplify $(I - A)(I + A + .. + A^{n-1})$.

7. Represent the following as the (i,j) entries in a matrix M:
   (i) $\sum_{k=1}^{n} a_{ki} b_{kj}$ (ii) $\sum_{p=1}^{n} b_{pj} \bar{a_{ip}}$ (iii) $\sum_{q=1}^{n} (B)_{qj} (A)_{qi}$

8. Give the (i,j) entries in $B^T A^T$ and $(AB)^T$. What do you notice ?

9. Give the (i,i) entry in AB and hence find Tr(AB). Repeat with BA. What do you notice ?

10. Let A $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{x}= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{y}= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Verify that $\mathbf{x}^T(A\mathbf{y}) = (\mathbf{x}^T A)\mathbf{y} = \mathbf{y}^T A^T \mathbf{x}$. Compute $\mathbf{x}^T\mathbf{y}$, $\mathbf{x}\mathbf{y}^T$, $Tr(\mathbf{x}\mathbf{y}^T)$ and Tr(A).

11. If $\mathbf{a} = [a_1, .., a_n]^T$ and $\mathbf{b} = [b_1, .., b_n]^T$, show that (i) $\mathbf{a}^T\mathbf{b} = (\mathbf{a}^T\mathbf{b})^T = \mathbf{b}^T\mathbf{a}$ (ii) $(\mathbf{a}\mathbf{b}^T)^T = \mathbf{b}\mathbf{a}^T$ (iii) $\mathbf{a}^T(\gamma\mathbf{b}) = \gamma(\mathbf{a}^T\mathbf{b})$ (iv) $\mathbf{e}_1^T(\gamma\mathbf{b}) = \gamma\mathbf{b}_i = \mathbf{b}^T(\gamma\mathbf{e}_1)$

12. If A $= [\mathbf{a}_1, \mathbf{a}_2, .., \mathbf{a}_n]$ and B $= \begin{bmatrix} \boldsymbol{\beta}_1^T \\ \vdots \\ \boldsymbol{\beta}_n^T \end{bmatrix}$, show that AB $= \mathbf{a}_1\boldsymbol{\beta}_1^T + .. + a_n\boldsymbol{\beta}_n^T$. Hence show that $(AB)^T = [\boldsymbol{\beta}_1, ..\boldsymbol{\beta}_n] \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \vdots \\ \boldsymbol{\alpha}_n^T \end{bmatrix} = B^T A^T$.

13. Show that $(AB)^* = B^* A^*$

14. Let $\mathbf{x} = [x_1, x_2]^T, \mathbf{y} = [y_1, y_2]^T$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
    (i) Write out $\mathbf{x}^T A\mathbf{y}$ and $\mathbf{x}^T B\mathbf{y}$
    (ii) Write out $\mathbf{x}^T A\mathbf{x}$ and $\mathbf{x}^T B\mathbf{x}$. What do you notice ?

(iii) Write out $\mathbf{x}^T A\mathbf{y}$ as a double sum
(iv) Compute M = $(A + A^T)/2$ and verify that $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T M\mathbf{x}$

15. (i) Show that $\mathbf{e}_i^T\mathbf{e}_j = \delta_{ij}$.
(ii) Show $E_{ij} = \mathbf{e}_i\mathbf{e}_j^T$
(iii) Hence show that $P^TP = I = PP^T$, for a permutation matrix P.

16. Write out $\sum_{i=1}^{m}\sum_{j=1}^{m} b_i m_{ij} c_j$ as a matrix product of 3 matrices

17. Show that A = $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$, where $E_{ij} = \mathbf{e}_i\mathbf{e}_j^T$. What happens if $\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}E_{ij} = 0$?

18. If P is n.s. and PA = H = H^2, show that APA = A.

19. Show that Tr(ABC) = Tr(BCA) = Tr(CAB). What scalar property did you use ?
Simplify $(ABCD)^T = ?$

20. TRUE OR FALSE:
(i) $(A^2)_{ij} = a_{ij}^2$, *(ii)* $Tr(A^2) = [Tr(A)]^2$ (iii) $A^2 - I = (A - I)(A + I)$ (iv) $A^2 - B^2 = (A + B)(A - B)$ (v) A$\mathbf{x}$= 2$\mathbf{x}$⇒ A = 2 (vi) A$\mathbf{x}$= 2$\mathbf{x}$⇒ A = 2I (vii) $\mathbf{x}^T A\mathbf{y} = \mathbf{y}^T A^T\mathbf{x}$.
Explain or give counter examples.

21. If A = $\mathbf{u}\mathbf{v}^T$ is n×n, with $\mathbf{u} \neq \mathbf{0} \neq \mathbf{v}$, show that A ≠ 0. Find Tr(A) and A^2.

22. Show that if A has a zero row so does AB and if Q has a zero column so does PQ.

23. Compute A^k, k = 1,2,.., if A = diag$(d_1, d_2, .., d_n)$

24. Show that Tr(AB) = Tr(BA) and hence that Tr(AB..YZ) = Tr(ZAB.. Y).

25. Compute A^k, k = 1,2,.., if A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

26. If A = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find matrices X and Y such that AXA = A and YAY = Y. Repeat with A = $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

27. If p(x) and q(x) are polynomials, show that p(A) and q(A) commute

28. If AB = BA, show that
(i) $A^kB = BA^k$ and hence that
(ii) $A^kB^r = B^rA^k$ for all k,r = 1,2,..
(iii) f(A)g(B) = g(B)f(A) for all polynomials f and g.
(iv) If in addition A is n.s. show that $A^{-k}B = BA^{-k}$, for all k = 1,2,..
(v) $Q^{-1}AQ$ and $Q^{-1}BQ$ commute also.

29. (i) If P$\mathbf{e}$= $\mathbf{e}$, show that $P^k\mathbf{e} = \mathbf{e}$ for all k = 1,2... ($\mathbf{e} = [1, 1, .., 1]^T$)
(ii) If $P \geq 0$ show that $P^k \geq 0$ for all k = 1,2,..
(iii) If P$\mathbf{e}$= $\mathbf{e}$, and $P \geq 0$ show that $(P^k)_{ij}/k \to 0$ as $k \to \infty$.
(iv) If $P\mathbf{x} = \lambda\mathbf{x}$ show that $P^k\mathbf{x} = \lambda^k\mathbf{x}$. Hence show that $\lambda^k/k \to 0$ as $k \to \infty$. What does this say about λ ?

30. If $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ compute $A^2 - 3A + 2I, B^3$ and p(A), where p(x) = $1 - x^2$.

31. If $(N)_{ij} = 0$ for $i \leq j$ what can you say about N^n ?

32. For any E, F and X, verify that:
X = EXF + (I - E)XF + EX(I - F) + (I - E)X(I - F)

33. Let B = EXF, C = (I - E)X(I - F), with $E^2 = E$ and $F^2 = F$. What happens to BC and CB ?

34. If A and B are symmetric (Hermitian) are AB and A + B also ?

35. If $|A|$ is mod A, i.e. $(\mid A \mid)_{ij} = \mid a_{ij} \mid$, show that:
(i) $\mid A \mid \geq 0$, (ii) $\mid A + B \mid \leq \mid A \mid + \mid B \mid$ (iii) $\mid \beta A \mid = \mid \beta \mid \mid A \mid$ (iv) $\mid AB \mid \leq \mid A \mid \mid B \mid$

36. If AB = BA compute $(A+B)^3, (A+B)^4, ..., (A+B)^k$

37. If $A > 0$ and $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ show that $A\mathbf{x} > \mathbf{0}$.

38. If $A\mathbf{x} = \lambda\mathbf{x}$, with $\lambda \in \mathbb{F}$, show that A must be square and that $A^2\mathbf{x} = \lambda^2\mathbf{x}$. Now find $A^k\mathbf{x}$, and hence show that $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$ for any polynomial p(λ).

39. Verify the previous question with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = 3$ and $p(x) = 2 - x^2$

40. Find all matrices that commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Repeat with $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

41. If A commutes with *all* 3×3 matrices , show that A = αI_3 for some α.

42. If AB = A + B, what is (A - I)(B - I) ? Hence show that AB = BA.

43. What can you say about the diagonal entries of a (skew) Hermitian matrix ? What are the diagonal entries of A^*A ? What is $Tr(A^*A)$? verify this with $\begin{bmatrix} 2+1 & i & 0 \\ 1 & 3-i & 4 \end{bmatrix}$

44. Let AXA = A and YAY = Y. Such matrices are called *inner* and *outer* inverses of A, and are denoted by A^- and $\hat{A}$ respectively.
(i) Show that AA^-, A^-A and $A\hat{A}, \hat{A}A$ are idempotent (ii) What about $I - AA^-$ and $I - A^-A$?

45. (i) If $A^T = -A$, show that $\mathbf{x}^TA\mathbf{x} = 0$ for *all* $\mathbf{x} \in \mathbb{F}^n$
(ii) If $\mathbf{x}^TA\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{F}^n$, show that $A^T - -A$ (*Hint*: use $\mathbf{e}_p, \mathbf{e}_p + \mathbf{e}_q$
(iii) If $\mathbf{x}^*A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, show that A = 0. (*Hint*: use $\mathbf{e}_p + i\mathbf{e}_q$)

46. (a) If $A^2 = A$ and $A^n = 0$ find A
(b) If $A^3 = A^2$ and $A^n = 0$ find A

47. (a) If B = PAQ and QP = I, find the sizes of the matrices. (b) Find B^2 and then B^k.

48. Let $\Gamma = \{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}; a, b \in \mathbb{R}\}$. Show that
(a) $\Gamma + \Gamma \subseteq \Gamma$ (b) $\alpha\Gamma \subseteq \Gamma$, *all* $\alpha \in \mathbb{R}$ (c) $\Gamma.\Gamma \subseteq \Gamma$ (d) $\Gamma^T = \Gamma$ (e) $A \in \Gamma \Rightarrow A^T A = diag(n, n) = AA^T$ where n = $a^2 + b^2$. (f) show that the map $\phi : \mathbb{C} \to \Gamma$ defined by $\phi(a + ib) = \{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\}$ is one-one and onto. (g) what else does ϕ preserve ?

49. If M = $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$ find M^k, k = 1,2,.. (induction!)

50. Show that $AA^T, A^T A, (B + B^T)$ are *always* symmetric for any matrix A and any $n \times n$ matrix B. What can you say about $(B - B^T)$?

51. Verify that B = $\frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T)$. What does it say ?

52. What can you say about AA^*, A^*A and $(B+B^*)/2, (B-B^*)/2$? Verify this for $\begin{bmatrix} 2+1 & i \\ 1 & 3-i \end{bmatrix}$

53. Show that $A^*A = -I$ is *never* possible for a complex matrix.

54. (a) Show that if A is (skew) symmetric so is $B^T AB$
(b) Show that if A is (skew) Hermitian, so is B^*AB

55. Show that every $n \times n$ matrix A has a *unique* decomposition:
$A = A_S + A_{SS}$, where $A_S^T = A_S$ and $A_{SS}^T = -A_{SS}$ (assume 2≠0)

56. (a) Show that every complex matrix has a unique decomposition M = A + iB, where A and B are real.
(b) Show that every complex matrix has a *unique* decomposition A = B + C, where $B^* = B$ and $C^* = -C$.

57. Show that every $n \times n$ matrix A has a unique decomposition A = L + D + U, where L and U^T are strictly lower triangular and D is diagonal.

58. Let A = $\begin{bmatrix} a & \mathbf{c}^* \\ 0 & D \end{bmatrix}$, where a = $(A)_{11}$. Compute $(AA^*)_{11}$ and $(A^*A)_{11}$. What happens if $AA^* = A^*A$?

59. Suppose that $AA^* = A^*A, U^* = U^{-1}, B = U^*AU$ and $C = A - \alpha I$. Also suppose that D is diagonal and E = βA with $\beta \neq 0$. Show that $BB^* = B^*B, CC^* = C^*C, DD^* = D^*D$ and $EE^* = E^*E$.

60. If $\{c_i, c_2, c_3, c_4\}$ are distinct, find a polynomial p(x) such that $p(c_i) = \bar{c}_i$ (the conjugate) i = 1,2,3,4.

61. True of False:
(i) $(A + B)^2 = A^2 + B^2$ (ii) $(A + B)^T = A^T + B^T$
(iii)$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A^k\mathbf{x} = \lambda^k\mathbf{x}$, (iv) If A ≠ 0, then Ref(A) ≠ 0.
(v) If A has 4 zero rows, then r(A) ≤ m-4, (vi) If A≠0 then r(A) ≥ 1
(vii) A is invertible iff N(A) ≠ (0). (viii) r(A + B) = r(A) + r(B)
(ix) If $P \geq 0$ then $P^k \geq 0$ (entry-wise). (x) If $[I_r, G]X = I$ then G must be absent.

62. True or False:
(i) $(A^T)_{ij} = a^T_{ij}$ (ii) If $B_{ij}ij = a^{-1}_{ij}$ then BA = I (iii) $Tr(A^2) = \sum a_{ii}$
(iv) $\mathbf{x}^T A = 2\mathbf{x}^T \Rightarrow A = 2I$ (iv) $\mathbf{x}^T 2 = \mathbf{x}\mathbf{x}$ (v) $\mathbf{x}^{-1} = 1/\mathbf{x}$
(vi) $(\mathbf{x}^T A\mathbf{x})^{-1} = 1/(\mathbf{x}^T A\mathbf{x})$ (vii) $(\mathbf{x}^T A\mathbf{x}) = \mathbf{x}^T(\frac{A+A^T}{2})\mathbf{x}$

63. Show that every 3×3 matrix is a sum of invertible matrices.

64. Verify that $\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} A & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$

65. If B is similar to A (i.e. $B = Q^{-1}AQ$), show that B + cI is similar to A + cI.

66. Why is $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ **similar** to $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$? Why is $\begin{bmatrix} I+AB & 0 \\ B & I \end{bmatrix}$ **similar** to $\begin{bmatrix} I & 0 \\ B & I+BA \end{bmatrix}$?

67. Verify that E = $Q\begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}Q^{-1}$ is idempotent for all X. Show that if $N^k = 0$ and $\begin{bmatrix} I & X \\ 0 & N \end{bmatrix}$ is idempotent then N must equal zero.

68. Verify that $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ iff AX - XD = C.

69. Show that $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = $ I.

70. If A commutes with A^* and $U^* = U^{-1}$, show that U^*AU and A + cI also commute with their "stars".

71. Let $\mathbf{u}, \mathbf{x} \in \mathbb{R}^n$.
(a) (i) If $\mathbf{u}^T\mathbf{x} > -1$ for all $\mathbf{x} > \mathbf{0}$, show that $\mathbf{u} \geq \mathbf{0}$ (ii) What if $\mathbf{u}^T\mathbf{x} > -a$ with $a > 0$?
(b) If $\mathbf{u}^T\mathbf{x} > 0$ for all $\mathbf{x} \underset{\neq}{\geq} \mathbf{0}$, show that $\mathbf{u} > \mathbf{0}$