MA 405 Matrix Problems
collected by R. Hartwig Jan 25 2005

2 Matrix Inversion

1. When is a diagonal matrix invertible ?

2. Verify that $\begin{bmatrix} a & c \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}cd^{-1} \\ 0 & d^{-1} \end{bmatrix}$.

3. If (I - AB)X = I, show that (I - BA)Y = I, where Y = I + BXA. Hence show that (I - AB) is invertible iff I - BA is invertible.

4. True of False:
(i) $(A+B)^{-1} = A^{-1} + B^{-1}$, (iii) A is invertible iff N(A) $\neq$ (0). (iii) If $[I_r, G]$ is invertible then G must be absent. (iv) $(A^{-1})_{ij} = a_{ij}^{-1}$ (v) $(kA)^{-1} = k^{-1}A^{-1}$

5. Verify that $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$. What happens if D = d is a scalar ?
Can you guess the inverse $\begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix}^{-1}$?

6. What is the inverse of $\begin{bmatrix} 0 & C \\ B & D \end{bmatrix}$ if any ?

7. If $N^k = 0$, show $(I-N)^{-1} = I + N + .. + N^{k-1}$
(ii) If $A^2 = cA$ with $c \neq 1$, show that $(I-A)^{-1} = I - A/(1-c)$
(iii) Show $A^2 = I$ iff $A^{-1} = A$ iff $(I-A)(I+A) = 0$.

8. If ABCD = I (all $n \times n$) find $(BC)^{-1}$

9. Show that $A^* = -A^{-1}$ is *never* possible for a complex matrix.

10. By inspection find the inverses (if any) of
(a) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ (d) E = $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
Draw the digraph associated with E and use it to compute E^4 and hence E^{-1}.

11. Let $E^2 = E$.
(i) Compute E(I - E) and E^k, k = 1,2,..
(ii) Show that $(I-E)^2 = I - E$ and hence give $(I-E)^k$
(iii) Simplify $[EX(I-E)]^2$
(iv) Show that $(I-2E)^2 = I$ and hence that $(I-2E)^{-1} = I - 2E$
(v) Given c $\neq -1$, find "d" such that (I + cE)(I + dE) = I. In particular what is $(I+E)^{-1}$?
(vi) If E is invertible then E = ?

12. Use counters to compute A^{-1} where A equals:

(i) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. If z = 1 - $\mathbf{b}^T A^{-1}\mathbf{c} \neq 0$ and $\zeta = A - \mathbf{c}\mathbf{b}^T$, show that $\zeta^{-1} = A^{-1} + A^{-1}\mathbf{c}\mathbf{b}^T A^{-1}/z$

14. If A = I + $\mathbf{u}\mathbf{v}^T$, verify that $A^{-1} = I - \mathbf{u}\mathbf{v}^T/(1 + \mathbf{v}^T\mathbf{u})$, given 1 + $\mathbf{v}^T\mathbf{u} \neq 0$

15. $(A - CB)^{-1} = A^{-1} + A^{-1}C(I - BA^{-1}C)^{-1}BA^{-1}$, provided I - $BA^{-1}C$ is invertible.

16. Do $\mathbf{u}^T = [1, 2, 3]$ and $\mathbf{u}$ have right inverses? left inverses ?

17. Verify that $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$. What about $\begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix}^{-1}$

18. What is the inverse of $\begin{bmatrix} 0 & C \\ B & D \end{bmatrix}$ if any ?

19. Use block matrices and induction to show that an upper triangular matrix is invertible iff all its diagonal entries are non-zero. What is $(T^{-1})_{ii}$?

20. If $P(\alpha) = \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$, show that:

    (i) $P(\alpha)P(\beta) = P(\alpha + \beta) = P(\beta)P(\alpha)$
    (ii) $P(-\alpha) = P(\alpha)^{-1} = P(\alpha)^T$. What is the meaning of $P(\alpha)$?

21. Compute $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$. What do you notice ? Repeat with $\begin{bmatrix} a & 0 & c \\ 0 & 1 & 0 \\ -c & 0 & a \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

22. If RAX = I_m, and R is n.s. show that AXR = I_m

23. If RA = $\begin{bmatrix} C \\ 0 \end{bmatrix}$ and $R^{-1} = [B, ?]$ show that A = BC.

24. Show that $(A^{-1})^T = (A^T)^{-1}$ and $(\bar{A})^{-1} = \overline{(A^{-1})}$. That is, transposition inversion and conjugation all commute.

25. (i) If A^{-1} and $(AB)^{-1}$ exists, give an expression for B^{-1} ?
    (ii) If $(AB)^{-1}$ and $(BA)^{-1}$ exist, giving expressions for A^{-1}, B^{-1}.

26. (i) If AB and BA are invertible, show that A and B also are.
    (ii) If A and AB are n.s. show that B is also n.s.
    (iii) If ABC, BCA and CAB are all n.s. , show that A, B and C also are n.s.

27. If B = $Q^{-1}AQ$ find $B^2, B^3, .., B^k$ and simplify. What is p(B) if p(x) is a polynomial in x ?

28. If AB = BA show that $A^{-1}B = BA^{-1}$, $A^{-1}B^{-1} = B^{-1}A^{-1}$ and $A^kB^r = B^rA^k$ for all k,r $\in \mathbb{Z}$.

29. Show that $A^2 = I$ iff $(A+I)(A-I) = I$

30. True or False:
(i) $(A^{-1})_{ij} = a_{ij}^{-1}$ (ii) $(AB)^{-1} = A^{-1}B^{-1}$
(iii) $\mathbf{x}^{-1} = 1/\mathbf{x}$ (iv) $(\mathbf{x}^T\mathbf{x})^{-1} = 1/(\mathbf{x}^T\mathbf{x})$
(v) $(cA)^{-1} = \frac{1}{c}A^{-1}$ (vi) $(A+B)^{-1} = A^{-1} + B^{-1}$
(vii) $(\overset{\bullet}{A})^{-1} = (A^{-1})^{\bullet}$. (viii) $Tr(A^{-1}) = [Tr(A)]^{-1}$.
(ix) If $A^5 = I$ then $(A^3)^{-1} = A^2$. (x) $(1-c)^{-1} = 1 - 1/c$

31. Show that any two of the following identities imply the third:
(i) $A^2 = I$ *(ii)* $A^T = A^{-1}$ *(iii)* $A^T = A$

32. If D $= diag(d_1, d_2, .., d_n)$, show that $D^{-1} = diag(1/d_1, .., 1/d_n)$.

33. Let PA = B and AQ = C, with P and Q invertible. Show that:
(i) A has a left inverse iff B does iff C does , (ii) A has a right inverse iff B does iff C does. (A is $m \times n$)

34. Show that the only invertible idempotent is I. What is the only nilpotent idempotent?

35. (i) Show that $\begin{bmatrix} C \\ 0 \end{bmatrix}$ cannot have a right inverse. (ii) Show using blocks, that $\begin{bmatrix} I_r & S \\ 0 & 0 \end{bmatrix}$ cannot have a left inverse, unless S is absent.

36. Show that A has a left inverse iff A^T has a right inverse.

37. Compute the inverses of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

38. If $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ and $R^{-1} = S = [S_1, S_2]$, where R_1 and S_1^T are $r \times n$, show that:
(i) $R_1S_1 = I_r, R_1S_2 = 0, R_2S_2 = I_{n-r}, R_2S_1 = 0$
(ii) $S_1R_1 + S_2R_2 = I_n$.
(iii) What happens if S = R^* as well ?

39. If a,b,c are distinct, show that the matrix $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is invertible.