Check: substitute into equation (1): 1(0) + 2(-1) + 3(1) = 1.

Note: (a) **always** add a multiple (=**multiplier**) of one row to another. (b) keep a list of the operations \( \rho_i + (\alpha) \rho_j \), that have been used. (c) in the tableaux **use arrows** with multipliers attached.

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
4 & 5 & 6 & 1 \\
7 & 8 & 10 & 2 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & -2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\rho_1 + (\alpha) \rho_2
\sim
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -3 & -6 & -3 \\
0 & -6 & -11 & -5 \\
\end{bmatrix}
\]

It is also convenient to "stack" the tableaux of equivalent systems.

In Case (a) we have **one** unique solution, in Case (b) **none**.

To "read off" the solutions in Case (c), we perform the following 4 steps:

Step(1) Switch off the X-ray machine and give the terminal equations. This gives

\[
x - z = -1 \text{ and } y + 2z = 1
\]

Step(2) Identify the pivot variables, and rewrite the terminal equations keeping the pivot variables **on the left**. The remaining non-pivot variables are brought over to the RHS. They become **free parameters**. In Case (c):
\[ x = -1 + z \] \text{ and } \[ y = 1 - 2z \]

Step (3) Substitute these variables into the column \( x \). In Case (c) \( x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + z \\ 1 - 2z \\ z \end{bmatrix} \)

Step (4) Collect each free variable into a separate column, and collect all constants into a separate column. In Case (c) we get

\[ x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = a + zb \]

Note: (i) \( z \) is a parameter which has to be assigned, e.g. \( z = 0 \) gives the particular solution \( x = [-1, 1, 0]^T \), while \( z = 1 \) gives the solution \( x = [0, -1, 1]^T \).

A line in 3D generally will look like \( a + zb \), for fixed vectors \( a \) and \( b \) and a real parameter \( z \in \mathbb{R} \).

To picture this line in three dimensions we must select a range of values for \( z \), such as \( z = 0, 0.1, 0.2,...,1 \), which can then be plotted. In Maple this would for example be:

\[ \text{plot3d}([-1+z,1-2z,z],z=0..1); \]

**WARNING:** the echelon matrix obtained at the end of Phase (I) *cannot* be used to read off the general solution and \( N(A) \). More elimination would be needed, which is better done by going through phase (II).