MA 405 Example 1, Jan 26, 2005

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ 7 & 8 & 10 & 2\end{array}\right] \underset{\rho_3+(-7)\rho_1}{\overset{\rho_2+(-4)\rho_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & \boxed{-3} & -6 & -3\\ 0 & -6 & -11 & -5\end{array}\right] \overset{\rho_3+(-2)\rho_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & -3 & -6 & -3\\ 0 & 0 & 1 & 1\end{array}\right] \overset{-\frac{1}{3}\rho_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & \boxed{1} & 1\end{array}\right] \underset{\rho_2+(-2)\rho_3}{\overset{\rho_1+(-3)\rho_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2\\ 0 & \boxed{1} & 0 & -1\\ 0 & 0 & 1 & 1\end{array}\right] \overset{\rho_1+(-2)\rho_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{array}\right] \text{ and } \left[\begin{array}{c} x\\ y\\ z\end{array}\right] = \left[\begin{array}{c} 0\\ -1\\ 1\end{array}\right].$$

Check: substitute into equation(1): 1(0) + 2(-1) + 3(1) = 1.

Note: (a) **always** add a multiple (=*multiplier*) of one row to another. (b) keep a list of the operations $\rho_i + (\alpha)\rho_j$, that have been used. (c) in the tableaux **use arrows** with multipliers attached. For example

$$\begin{array}{c} \\ \boxed{-4}\rightarrow \\ -7 \rightarrow \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ 7 & 8 & 10 & 2\end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & \boxed{-3} & -6 & -3\\ 0 & -6 & -11 & -5\end{array}\right]$$

It is also convenient to "stack" the tableaux of equivalent systems.

Case (a):

$$\begin{array}{r} \\ -4 \rightarrow \\ -7 \rightarrow \\ \\ \\ -2 \rightarrow \\ \\ -\frac{1}{3}\times \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 1 & \\ 4 & 5 & 6 & 1 & \\ 7 & 8 & \textcircled{10} & \textcircled{2} & \\ \hline 1 & 2 & 3 & 1 & \\ 0 & \boxed{-3} & -6 & -3 & \\ 0 & -6 & -11 & -5 & \\ \hline 1 & 2 & 3 & 1 & \\ 0 & -3 & -6 & -3 & \\ 0 & 0 & 1 & 1 & \\ \hline 1 & 2 & 3 & 1 & \leftarrow -3 \\ 0 & 1 & 2 & 1 & \leftarrow -2 \\ 0 & 0 & \boxed{1} & 1 & \\ \hline 1 & 2 & 0 & -2 & \leftarrow -2 \\ 0 & \boxed{1} & 0 & -1 & \\ 0 & 0 & 1 & 1 & \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & -1 & \\ 0 & 0 & 1 & 1 & \end{array}\right]$$

Case (b):

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ 7 & 8 & \textcircled{9} & 2\\ \hline 1 & 2 & 3 & 1\\ 0 & \boxed{-3} & -6 & -3\\ 0 & -6 & -12 & -5\\ \hline 1 & 2 & 3 & 1\\ 0 & -3 & -6 & -3\\ 0 & 0 & 0 & 1\end{array}\right]$$

Case (c):

$$\begin{array}{r} \\ -7\ \boxed{-4} \rightarrow \\ \rightarrow \\ \\ -\frac{1}{3}\times \\ -2 \rightarrow \\ \rightarrow \\ -2 \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1\\ 4 & 5 & 6 & 1\\ 7 & 8 & 9 & \textcircled{1}\\ \hline 1 & 2 & 3 & 1\\ 0 & \boxed{-3} & -6 & -3\\ 0 & -6 & -12 & -6\\ \hline 1 & 2 & 3 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0\\ \hline \boxed{1} & 0 & -1 & -1\\ 0 & \boxed{1} & 2 & 1\\ \hline 0 & 0 & 0 & 0\end{array}\right]$$

In Case (a) we have **one** unique solution, in Case (b) **none**.

To "read off" the solutions in Case (c), we perform the following 4 steps:

Step(1) Switch off the X-ray machine and give the terminal equations. This gives

$$\text{x -z = -1 and y + 2z = 1}$$

Step(2) Identify the pivot variables, and rewrite the terminal equations keeping the pivot variables **on the left**. The remaining non-pivot variables are brought over to the RHS. They become **free parameters**. In Case (c):

x = -1 + z and y = 1 - 2z

Step(3) Substitute these variables into the column **x**. In Case (c) $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+z \\ 1-2z \\ z \end{bmatrix}$

Step(4) Collect each free variable into a separate column, and collect all constants into a separate column. In Case (c) we get

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \mathbf{a} \ + z\mathbf{b}$$

Note: (i) z is a parameter which has to be assigned, e.g. z = 0 gives the particular solution **x**= $[-1, 1, 0]^T$, while z = 1, gives the solution **x**= $[0, -1, 1]^T$.

A line in 3D generally will look like **a** + z**b**, for fixed vectors **a** and **b** and a real parameter $z \in \mathbb{R}$.
To picture this line in three dimensions we must select a range of values for z, such as z = 0, 0.1, 0.2,..,1, which can then be plotted. In **maple** this would for example be :

```
>plot3d([-1+z,1-2z,z],z=0..1);
```

WARNING: the echelon matrix obtained at the end of Phase (I) *cannot* be used to read off the general solution and N(A). More elimination would be needed, which is better done by going through phase (II).