Problem 1 Consider the function \( f(x) = \frac{1}{1+x^2} \) in \([-5, 5]\).

1. What can be said about the approximation of \( f \) over \([-5, 5]\) by Taylor polynomials about \( x = 0 \)?

2. Let \(-5 = x_0 < x_1 < \ldots < x_n = 5\) be \( n + 1 \) equally spaced nodes on \([-5, 5]\). For \( n = 5 \) and 10, find \( p_n \) the polynomial of degree \( n \) agreeing with \( f \) at the points \( x_0, x_1, \ldots, x_n \). Plot \( p_5, p_{10} \) and \( f \) together.

You can use the MATLAB functions \texttt{polyfit} and \texttt{polyval} to do that. Example

```matlab
x = linspace(-5,5,11); % creates nodes vector (n=10)
f = 1./(1+x.^2); % f at the nodes
xf = linspace(-5,5,200); % "fine mesh"
ff = 1./(1+x.^2); % f on the fine mesh
p = polyfit(x,f,10); % construction of the polynomial
pf = polyval(p,xf); % evaluation of p on fine mesh
plot(x,f,'ro',xf,ff,xf,pf); % plot everything
```

3. What can you say about \( f - p_n \) as \( n \to \infty \)?

Problem 2 Consider a family of orthogonal polynomials with respect to \((\cdot, \cdot)_w\), assuming that the leading coefficient of each polynomial is normalized to 1.

1. If we set \( p_{-1}(x) = 0 \) and \( p_0(x) = 1 \), show by induction that the following three term recursive formula holds

\[
p_{k+1} = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x).
\]

Give formulas for \( \alpha_k \) and \( \beta_k \).

2. On the interval \([a, b] = [-1, 1]\) and with the weight function \( w \equiv 1 \), use the Gram-Schmidt process to construct \( p_i \), \( i = 0, \ldots, 3 \) (first four Legendre polynomials).

3. On the interval \([a, b] = [-1, 1]\) and with the weight function \( w(x) = \frac{1}{\sqrt{1-x^2}}\), use the Gram-Schmidt process to construct \( p_i \), \( i = 0, \ldots, 3 \) (first four Chebyshev polynomials).

4. Plot a few of those polynomials (both Legendre and Chebyshev).

Problem 3 1. Show that the Chebyshev polynomials satisfy

\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), n \geq 2.
\]

2. For \(|x| \leq 1\), prove that \( T_n(x) = \cos(n \arccos x) \).

3. Check that \( T_n \) has \( n \) zeros in \([-1, 1]\) at \( x_k = \cos(\tfrac{k\pi}{n}) \), \( k = 1, 2, \ldots, n \).

4. Check that \( T_n \) has \((n+1)\) extremas in \([-1, 1]\) at \( x_k' = \cos(\tfrac{k\pi}{n}) \), \( k = 0, 1, \ldots, n \), where \( T_n(x_k') = (-1)^k \).

5. (BONUS) Let \( \tilde{P}_n \) be the set of polynomials of degree \( n \) with leading coefficient 1 (coefficient of \( x^n = 1 \)). Prove the optimality property of the Chebyshev polynomials, i.e.

\[
\max_{-1\leq x \leq 1} \frac{|T_n(x)|}{2^{n-1}} \leq \max_{-1\leq x \leq 1} |p(x)|, \quad \forall p \in \tilde{P}_n.
\]

Hint: assume by contradiction that there exists \( p \in \tilde{P}_n \) with \( \max_{-1\leq x \leq 1} |p(x)| < \frac{1}{2^{n-1}} \) and look at \( r = \frac{T_n}{2^{n-1}} - p \).

6. For general interpolation, it is possible to show

\[
f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0) \ldots (x-x_n).
\]

Determine the best way to place the interpolation nodes \( x_0, x_1, \ldots, x_n \) in a general interval \([a, b]\).

Hint: consider the change of variable \( \bar{x} = \frac{1}{2}[(b-a)x + a + b] \).

7. Redo Problem 1, point 2. using the "optimal" nodes.