8/25/09: Graph Theory

Goal: review/introduce concepts from graph theory

A graph is an ordered pair $G = (V, E)$ where

$V$: set of vertices/nodes; $|V|$ = # vertices

$E$: set of edges; $|E|$ = # edges

If two vertices $v$ and $v'$ have an edge between them, they are said to be adjacent.

Graphs can be directed or not.

Examples:

```
(1) Jim ↔ Jill

(2) Jack — Jane

(3) "have corresponded"

(4) Jim → Jill

(5) Jack ← Jane

(6) "like"

(7) Undirected

(8) Directed
```
The above relations can be described by adjacency matrices:

\[ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

indirected

(directed: symmetric)

A notion of degree can be defined.

For undirected,

\[ \deg_i = k_i = \sum_{j=1}^{V} a_{ij} \]

ex: \( k_4 = 3 \)

For directed,

\[ \begin{align*}
\deg_i^{\text{in}} &= \sum_{j=1}^{V} a_{ij} \\
\deg_i^{\text{out}} &= \sum_{j=1}^{V} a_{ji} 
\end{align*} \]

ex: \( \deg_4^{\text{in}} = 1 \), \( \deg_4^{\text{out}} = 2 \)
Important graphs: complete graphs.
Each vertex is connected to each other.

\[ K_1, K_2, K_3, K_4, K_5 \]

A planar graph is a graph that can be embedded in the plane. In other words, it can be drawn in the plane in such a way that its edges intersect only at their endpoints.

Examples: \( K_1, K_2, K_3 \) are planar.
\( K_4, K_5 \) are not.

Discuss Problem 1 of Project 1.

Incidence matrices can also be used to describe networks.

For a given network \( C \) has size \(|V| \times |E|\).
If node \( i \) is an endpoint for edge \( j \), then \( C(i,j) = 1 \).
Otherwise, \( C(i,j) = 0 \).

Note: need to number edges as well.
For a directed graph in the column of \( C \) there is 1 for one of the vertices and -1 for the other.

Back to previous example

\[
C = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & -1 & 1
\end{bmatrix}
\]

undirected

directed