Definition: In each of the problems we use the following:

1. $f(E) = \{f(x) \mid x \in E\}$ and
2. $f^{-1}(F) = \{x \mid f(x) \in F\}$

where $E \subseteq X$ and $F \subseteq Y$.

1. Suppose $f : X \to Y$ and that $A \subseteq X$.

   To show that $A \subseteq f^{-1}(f(A))$, assume that $x \in A$. By 1 in the above definition, this means that $f(x) \in f(A)$. However by 2 above, this means that $x \in f^{-1}(f(A))$. This concludes the inclusion portion of the problem.$\blacksquare$

   Now suppose that the two are equal for all A, we will show that f must be injective. Suppose otherwise and that $a, b \in X$, with $a \neq b$, and $f(a) = f(b)$. Let $A = \{a\}$. In this case $f(A) = \{f(a)\}$, and $f^{-1}(f(A)) = \{a, b\} \neq A$. This contradiction implies that f must be one-to-one.$\blacksquare$

   So suppose that f is one-to-one. We will show that the two sets are equal. We need only show that $f^{-1}(f(A)) \subseteq A$. So let $x \in f^{-1}(f(A))$. Again using part 2 of the above definition, $x \in f^{-1}(f(A))$ implies that $f(x) \in f(A)$. If this is true $f(x) = f(y)$ for some $y \in A$. Since f is one-to-one, $x = y \in A$, and we are done.$\blacksquare$

2. For this problem we must start by proving $f(f^{-1}(B)) \subseteq B$ if $B \subseteq Y$. To this end let $y \in f(f^{-1}(B))$. This means that there is an $x \in f^{-1}(B)$ so that $f(x) = y$. But by the definition part 2, $y = f(x) \in B$.$\blacksquare$

   As before it is easy to show that if $f(f^{-1}(B)) = B$, for all $B \subseteq Y$, then f must be onto. Suppose f is not onto. Then there is a $y \in Y$ which is not the image of any x in X. For this y let $B = \{y\}$. Since nothing maps onto y, $f^{-1}(B) = \emptyset$. Thus $f(f^{-1}(B)) = \emptyset \neq \{y\} = B$.

   So suppose that f is onto (surjective). Let $y \in B$. Since f is onto, there is an $x \in X$ so that $f(x) = y$. In other words, an x so that $x \in f^{-1}(\{y\}) \subseteq f^{-1}(B)$. This means that $y = f(x) \in f(f^{-1}(B))$. Hence $B \subseteq f(f^{-1}(B))$.

The next two exercises are very similar to these two, except perhaps easeier.