Test 1

FEBRUARY 27, 2008

MA 537 Name ____________________

1. (20%) Consider the family of maps $f_\lambda(x) = \lambda - \frac{1}{x}$.

   (a) Find the equation in the (λ, x) variables for the curve of fixed points of $f_\lambda(x)$ and sketch this curve in the (λ, x) - plane.

   (b) There are two Tangent Bifurcations. Find the λ and x values where they occur.

   (c) State the hypothesis of the Tangent Bifurcation Theorem and show that they are all satisfied at each of the points you found in part b.

2. (20%) Apply the Pitchfork Bifurcation Theorem to show that the family of maps $f_\lambda(x) = \lambda \sin(x)$ undergoes a pitchfork bifurcation when $\lambda = 1$ and $x = 0$. Draw the bifurcation curve in the (λ, x)-plane near $(\lambda, x) = (1, 0)$.
   diagram

3. (15%) Find the solution to the initial value problem:

Drop

$$P(n) - 3P(n-1) = 5^n; \qquad P(0) = 1.$$

4. (15%) Consider the function f defined by:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < 0.5 \\ 2 - 2x & \text{if } 0.5 \le x < 1 \end{cases}$$

   (a) Graph f, f^2 and f^3. How many points are fixed under each of these maps?

   (b) How many points of prime period 3 does f have?

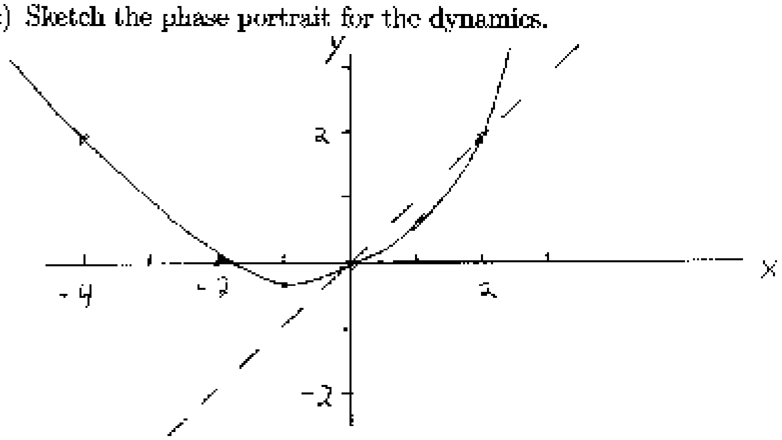
5. (15%) Find the fixed points of $f(x) = x + 2x^3 + x^4$ and decide if they are attracting, repelling, or neutral. If a fixed point is neutral, decide if it is weakly attracting, weakly repelling or neither.

6. (15%) This problem is about the dynamics of the system whose function has the graph below.

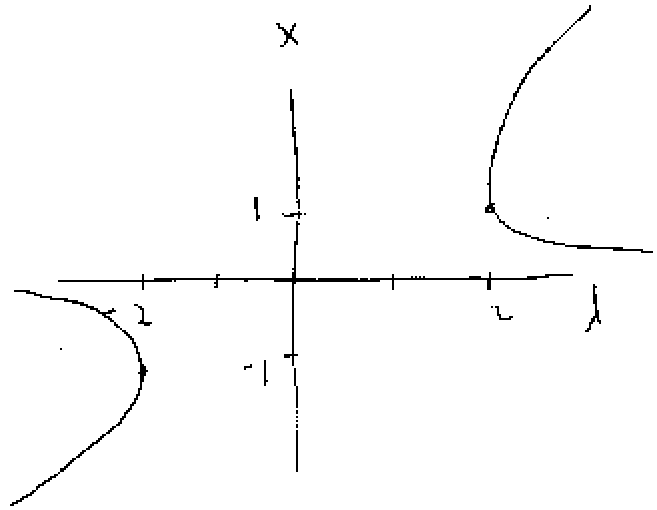
   (a) What are the fixed points? Which are attracting and which are repelling?

   (b) What are the eventually fixed points?

   (c) Sketch the phase portrait for the dynamics.



$f_\lambda(x) = \lambda - \frac{1}{x} = x$



$\lambda = x + \frac{1}{x}$

$\frac{d\lambda}{dx} = 1 - \frac{1}{x^2} = 0$

$x^2 = 1$

$x = \pm 1$

$(\lambda, x) = (2, 1)$ & $(-2, -1)$

$f_\lambda(x)$ is C^2,

$f_{\lambda_0}(x_0) = x_0$ $\qquad f_2(1) = 2 - \frac{1}{1} = 1$ $\qquad f_{-2}(-1) = -2 - \frac{1}{-1}$

$f'_{\lambda_0}(x_0) = 1$ $\qquad f'_\lambda(x) = + \frac{1}{x^2}$ $\qquad f'_2(1) = 1,$ $\qquad f'_{-2}(-1) = 1$

$f''_{\lambda_0}(x_0) \neq 0$ $\qquad f''_\lambda(x) = -\frac{2}{x^3}$ $\qquad f''_2(1) = -2$ $\qquad f''_{-2}(-1) = -2$

$\frac{\partial}{\partial \lambda} f_\lambda(x_0)\Big|_{\lambda=\lambda_0} \neq 0$ $\qquad \frac{\partial}{\partial \lambda} f_\lambda(x_0)\Big|_{\lambda_0} = 1$ always

$f_\lambda(x) = \lambda \sin x$

$f_\lambda(0) = \lambda \sin 0 = 0$ ✓

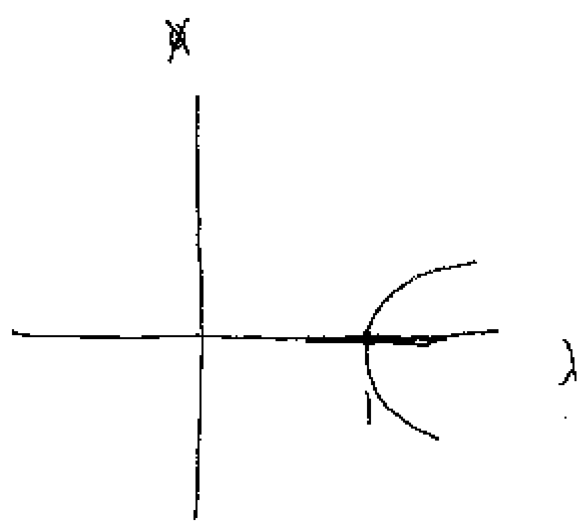
$f'_\lambda(x) = \lambda \cos x$ $\qquad f'_1(0) = 1 \cos 0 = 1$ ✓

$f''_\lambda(x) = -\lambda \sin x$ $\qquad f''_1(0) = -1 \cdot \sin 0 = 0$ ✓

$f'''_\lambda(x) = -\lambda \cos x$ $\qquad f'''_1(0) = -1 \cos 0 = -1 \neq 0$

$$\frac{\partial}{\partial \lambda} f'_\lambda(x) = \frac{\partial}{\partial \lambda} \left(\lambda \cos x \right) = \cos x$$

$$\frac{\partial}{\partial \lambda} f'_\lambda(0) \Big|_{\lambda = 1} = \cos 0 = 1 \neq 0$$



3. $P(n) - 3P(n-1) = 5^n \qquad P(0)$

$$P(n) - 3P(n-1) = 0$$

$$P_c(n) = A\ 3^n \qquad \text{complementary}$$

$$P_p(n) = B\,5^n$$

$$B\,5^n - 3\cdot B\,5^{n-1} = 5^n$$

$$5B - 3B = 5$$

$$2B = 5$$

$$B = \frac{5}{2}$$

$$P(n) = \frac{5}{2}\cdot 5^n + A\cdot 3^n$$

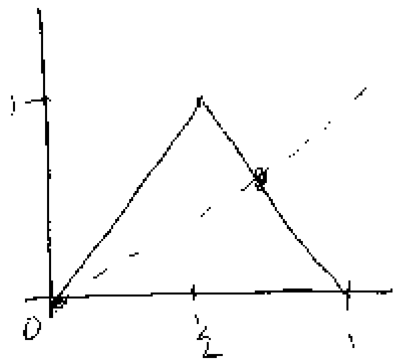
$$P(0) = 1 = \frac{5}{2} + A$$

$$A = 1 - \frac{5}{2} = -\frac{3}{2}$$

$$P(n) = \frac{5}{2}\cdot 5^n - \frac{3}{2}\cdot 3^n$$

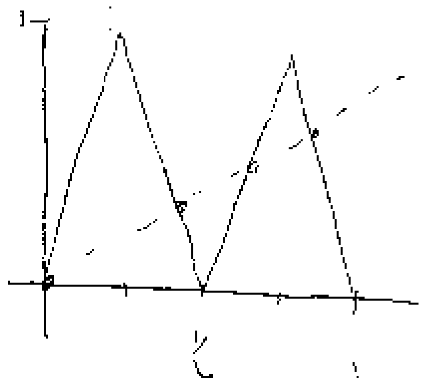
4.

f



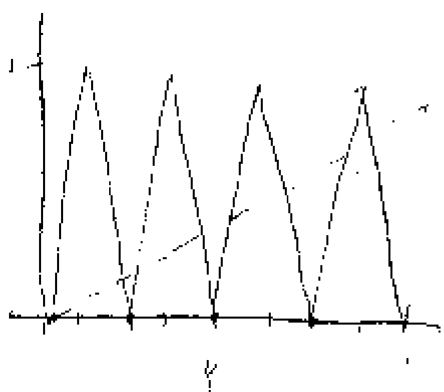
2 fixed points

f^2



4 fixed points

f^3



8 fixed points

2 of these are fixed for f.

6 prime period 3 points.

5. $f(x) = x + 2x^3 + x^4 = x$

$2x^3 + x^4 = 0$

$x^3(2+x) = 0$

$x = 0$ & $x = -2$ are the fixed points

$f'(x) = 1 + 6x^2 + 4x^3$

$f'(0) = 1$ neutral.

$f'(-2) = 1 + 24 - 32 = -7 < -1$ repelling

$f''(x) = 12x + 12x^2$

$f''(0) = 0$

$f'''(x) = 12 + 24x$

$f'''(x) = 12 > 0$ weakly repelling

6. 0 & 2 are fixed.

2 is repelling

0 is attracting

-2 & -4 are eventually fixed