MA 426 Test 2

APRIL 23, 2010

Dr. Franke Name ____________________

1. (10%) Let $f : \Re^n \to \Re^m$. Define f is differentiable at x_o.

2. (10%) State the Inverse Function Theorem.

3. (20%) Give a counter example for each statement.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and K is a connected subset of $\mathbb{R}$ then $f^{-1}(K)$ is connected.

   (b) If $f : \Re^1 \to \Re^2$ is differentiable, then there exists $c \in (0,1)$ such that
   $$f(1) - f(0) = Df(c) \cdot (1 - 0).$$

4. (13%) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on all of $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is bounded then $f(A)$ is bounded.

5. (20%) Let $A = \{(x, y) \in [0,1] \times [0,1] : x \text{ and } y \text{ are both rational}\}$.

   (a) Does A have area (2-dimensional volume)? (explain)

   (b) Is A a set of measure zero in $\Re^2$? Prove your answer.

6. (27%) Let $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   (a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$. (You might need to take a limit.)

   (b) Does f have a directional derivative at $(0,0)$ in the direction $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$? Find it or show it doesn't exist.

   (c) Is f differentiable at $(0,0)$? explain.

1. Let $f: R^n \to R^m$. f is differentiable at x_0 iff $\exists$ a linear map $Df_{x_0} : R^n \to R^m \ni$

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df_{x_0} (x-x_0) \|}{\| x - x_0 \|} = 0$$

2. Let $A \subset R^n$ be open and let $f: A \to R^m$ be of class C^1

Let $x_0 \in A$ and $\det(Df_{x_0}) \neq 0$. Then $\exists$ open neighborhood U of x_0 and an open neighborhood V of $f(x_0)$ such that

$f|_U$ is 1-1 onto V. The inverse $f^{-1}: V \to U$ is C^1 and

$$Df^{-1}_{(f(x_0))} = \left(Df_{x_0}\right)^{-1}$$

3a. $f(x) = x^2$ $\quad K = [1, 4]$, $\quad f^{-1}(K) = [-2, -1] \cup [1, 2]$

b. $f(x) = (x^2, x^3)$, $\quad Df = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}$

$f(1) - f(0) = (1,1) - (0,0) = (1,1)$

$Df_c (1-0) = \begin{pmatrix} 2c \\ 3c^2 \end{pmatrix} \cdot 1 = (1,1)$

$2c = 1$ & $3c^2 = 1$

$c = \frac{1}{2}$ but $3\left(\frac{1}{2}\right)^2 = \frac{3}{4} \neq 1$.

4. Let $f : R^n \to R^m$ be continuous on R^n. Let $A \subset R^n$ be bounded. Then $cl\,A$ is closed and bounded.

$cl\,A$ is compact. $f(cl\,A)$ is compact. Thus

$f(A) \subset f(cl\,A)$ is bounded.

5. a. A does not have volume 1_A is not integrable.

For any partition P of $[0,1] \times [0,1]$

$U(1_A, P) = 1$ and $L(1_A, P) = 0$.

Thus $\overline{\int} 1_A = 1$ and $\underline{\int} 1_A = 0$.

Thus 1_A is not integrable.

b. A has measure zero. A is a countable set.

So we can number them ~~$x_1, x_2, \ldots$~~ $(x_1, y_1)\ (x_2, y_2) \ldots$

Let $U_{(x_i, y_i)} = \left(x_i - \frac{\sqrt{\varepsilon}}{2^{i+2}}, x_i + \frac{\sqrt{\varepsilon}}{2^{i+2}} \right) \times \left(y_i - \frac{\sqrt{\varepsilon}}{2^{i+2}}, y_i + \frac{\sqrt{\varepsilon}}{2^{i+2}} \right)$. These cover A

$v\left(U_{(x_i, y_i)}\right) = 2 \cdot \frac{\sqrt{\varepsilon}}{2^{i+2}} \cdot 2 \frac{\sqrt{\varepsilon}}{2^{i+2}} = \frac{\varepsilon}{2^{2i+2}} < \frac{\varepsilon}{2^{i+1}}$

$\sum_{i=1}^{\infty} v\left(U_{(x_i, y_i)}\right) < \varepsilon$.

6,

a. $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

b. $$\lim_{t\to 0} \frac{f\left((0,0)+t\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right)-f(0,0)}{t} = \lim_{t\to 0} \frac{\dfrac{\frac{t^3}{\sqrt{2}\cdot 2}}{2t^2\cdot\frac{1}{2}}}{t} =$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{2}\cdot 2} = \frac{1}{2\cdot\sqrt{2}} \neq 0$$

c. If f is differentiable $Df_{(0,0)} = (0 \quad 0)$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy^2}{x^2+y^2} - 0 - (0,0)\begin{pmatrix}x\\y\end{pmatrix}}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}$$

on $x=y$ $$\lim_{x\to 0} \frac{x^3}{(2x^2)^{3/2}} = \lim_{x\to 0} \frac{1}{2^{3/2}} \neq 0.$$ Not differentiable.