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3. Let L be a linear map of $R^n \to R^m$, let $g: R^n \to R^m$ be such that $\| g(x) \| \le M \|x\|^2$, and let $f(x) = L(x) + g(x)$. Prove that

$Df(0) = L$.

Proof: $0 \le \dfrac{\| f(x) - f(0) - L(x-0) \|}{\|x - 0\|} = \dfrac{\| L(x) + g(x) - L(0) - g(0) - L(x-0) \|}{\|x - 0\|}$.

Since L is linear $L(0) = 0$.

at $x = 0$, $\| g(0) \| \le M \|0\|^2 = 0$ so $g(0) = 0$.

Thus $\dfrac{\| L(x) + g(x) - L(0) - g(0) - L(x-0) \|}{\| x - 0 \|} = \dfrac{\| L(x) + g(x) - L(x) \|}{\|x\|}$

$= \dfrac{\| g(x) \|}{\|x\|} \le M \dfrac{\|x\|^2}{\|x\|} = M \|x\|$.

Since $\lim_{x \to 0} M \|x\| = 0$, the squeeze theorem says

$\lim_{x \to 0} \dfrac{\| f(x) - f(0) - L(x-0) \|}{\| x - 0 \|} = 0$ so $Df(0) = L$.

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5. Find the tangent vector to the curve $c(t) = (3t^2, e^t, t+t^2)$

at the point corresponding to $t=1$

$$\frac{d\,c(t)}{dt} = (6t, e^t, 1+2t)$$

$$\left.\frac{d\,c(t)}{dt}\right|_{t=1} = (6, e, 3)$$

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1. Use theorem 6.4.1 to show that $f(x,y)$ defined by

$$f(x,y) = \begin{cases} \dfrac{(xy)^2}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

We need to show that the partials are continuous.

$$\frac{\partial f}{\partial x} = \frac{\sqrt{x^2+y^2}\;2xy^2 - (xy)^2 \dfrac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{2xy^2}{\sqrt{x^2+y^2}} - \frac{x^3y^2}{(x^2+y^2)^{3/2}} \quad \text{if } (x,y)\neq(0,0)$$

This is clearly continuous away from $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

P.344 1. continued

$\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ needs to be shown to be 0.

$$\left|\frac{\partial f}{\partial x}\right| = \left|\frac{2xy^2}{\sqrt{x^2+y^2}} - \frac{x^3y^2}{(x^2+y^2)^{3/2}}\right| \le \frac{(x^2+y^2)|y|}{\sqrt{x^2+y^2}} + \frac{(x^2+y^2)^2|x|}{(x^2+y^2)^{3/2}}$$

$$= \sqrt{x^2+y^2}\,|y| + \sqrt{x^2+y^2}\,|x|$$

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,|y| + \sqrt{x^2+y^2}\,|x| = 0$$

So by the squeeze theorem.

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = 0.$$

Thus $\frac{\partial f}{\partial x}$ is continuous at $(0,0)$.

By the symmetry of x & y in the problem, $\frac{\partial f}{\partial y}$ is also continuous at $(0,0)$. Thus f is differentiable.

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5. Find a function $f: R^2 \to R$ that is differentiable at each point but whose partials are not continuous at $(0,0)$.

There are many possible examples.

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{if neither } x \text{ nor } y \text{ is zero } \cancel{x \neq 0 \text{ and } y} \\ x^2 \sin\frac{1}{x} & \text{if } y = 0 \text{ and } x \neq 0 \\ y^2 \sin\frac{1}{y} & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x} = \begin{cases} 2x \sin\frac{1}{x} - \frac{x^2}{x^2}\cos\frac{1}{x} & \text{if neither } x \text{ nor } y \text{ is zero} \\ 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } y = 0 \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For the 2 zeros

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h \to 0} \frac{\left(h^2 \sin\frac{1}{h} + y^2 \sin\frac{1}{y}\right) - y^2 \sin\frac{1}{y}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h}}{h} = 0$$

similarly

$$\frac{\partial f}{\partial y} = \begin{cases} 2y\sin\frac{1}{y} - \cos\frac{1}{y} & \text{if neither } x \text{ nor } y \text{ is zero} \\ 2y\sin\frac{1}{y} - \cos\frac{1}{y} & \text{if } x=0 \text{ and } y\neq 0 \\ 0 & \text{if } y=0 \text{ and } x\neq 0 \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

$\lim\limits_{(x,y)\to(0,0)} 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ oscilates between ± 1, does not exist.

$\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are not continuous at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{\| f(x,y) - f(0,0) - Df_{(0,0)}(x-0, y-0)\|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\| x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y} - 0 - (0,0)\binom{x}{y}\|}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{|x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y}|}{\sqrt{x^2+y^2}}$$

now $\dfrac{|x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y}|}{\sqrt{x^2+y^2}} \le \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$

Since $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$. The squeeze theorem gives

$\lim\limits_{(x,y)\to(0,0)} \dfrac{|x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y}|}{\sqrt{x^2+y^2}} = 0$. So f is differentiable at $(0,0)$.

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2. Verify the chain rule for

$$u(x,y,z) = xe^y, \qquad v(x,y,z) = (\sin x)\, yz$$

and $\qquad f(u,v) = u^2 + v \sin u$

with $\quad h(x,y,z) = f(u(x,y,z),\ v(x,y,z))$

Here we have $R^3 \to R^2 \to R^1$. The chain rule says

$$Dh = Df \cdot Dg \qquad \text{where} \quad g(x,y,z) = (xe^y,\ (\sin x)yz)$$

$$Dg = \begin{pmatrix} e^y & xe^y & 0 \\ (\cos x)yz & (\sin x)z & (\sin x)y \end{pmatrix}$$

$$Df = \begin{pmatrix} 2u + v\cos u & \sin u \end{pmatrix}$$

$$Df \cdot Dg = \begin{pmatrix} 2u + v\cos u & \sin u \end{pmatrix} \begin{pmatrix} e^y & xe^y & 0 \\ (\cos x)yz & (\sin x)z & (\sin x)y \end{pmatrix}$$

$$= \Big((2u + v\cos u)e^y + (\sin u)(\cos x)yz \qquad (2u + v\cos u)xe^y + (\sin u)(\sin x)z \qquad (\sin u)(\sin x)y \Big)$$

$$= \Big(\left[2xe^y + (\sin x)yz\cos(xe^y)\right]e^y + \sin(xe^y)(\cos x)yz \qquad \left[2xe^y + (\sin x)yz\cos(xe^y)\right]xe^y + \sin(xe^y)(\sin x)z \qquad \sin(xe^y)(\sin x)\,y \Big)$$

2 continued

$$h(x,y,z) = \left((xe^y)^2 + (\sin x) yz \sin(xe^y) \right)$$

$$\frac{\partial h}{\partial x} = \cancel{2(xe^y)e^y + (\cos x) yz + \cos(xe^y)\cdot e^y}$$

$$\frac{\partial h}{\partial x} = 2(xe^y)e^y + yz\left[\cos x \sin(xe^y) + (\sin x)\cos(xe^y)\cdot e^y\right]$$

$$\frac{\partial h}{\partial y} = 2(xe^y)xe^y + (\sin x) z\left[\sin(xe^y) + y\cos(xe^y)\cdot xe^y\right]$$

$$\frac{\partial h}{\partial z} = (\sin x)\, y \sin(xe^y)$$

These partials check with the matrix on the other page.

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5. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable with A convex, and suppose $\|\text{grad } f(x)\| \le M$ for $x \in A$. Prove $|f(x) - f(y)| \le M\|x - y\|$ for all $x, y \in A$. Do you think this is true if A is not convex? $\boxed{\text{NO!}}$

Proof: Let $x, y \in A$. Since A is convex we can use the Mean Value Theorem to at a point c on the line segment between x & y where

$$f(x) - f(y) = Df(c) \cdot (x - y)$$

$$|f(x) - f(y)| \le \|Df(c)\| \cdot \|x - y\| = \|\text{grad } f(c)\| \cdot \|x - y\|$$

$$\le M \|x - y\| \quad \text{since } c \in A.$$

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1. If $f : A \subset R^n \to R^m$ and $g : B \subset R^n \to R^m$ are differentiable functions on the (open) sets A and B and α, β are constants, prove that $\alpha f + \beta g : A \cap B \subset R^n \to R^m$ is differentiable and

$D(\alpha f + \beta g)(x) = \alpha Df(x) + \beta Dg(x)$.

Proof: Let $x_0 \in A \cap B$. Since both f and g are differentiable at x_0,

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|}{\| x - x_0 \|} = 0 \quad \& \quad \lim_{x \to x_0} \frac{\| g(x) - g(x_0) - Dg(x_0)(x - x_0) \|}{\| x - x_0 \|} = 0$$

Thus $$\lim_{x \to x_0} |\alpha| \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|}{\| x - x_0 \|} + |\beta| \frac{\| g(x) - g(x_0) - Dg(x_0)(x - x_0) \|}{\| x - x_0 \|} = 0$$

Since $$0 \le \frac{\| (\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0) - (\alpha Df(x_0) + \beta Dg(x_0))(x - x_0) \|}{\| x - x_0 \|}$$

$$\le \frac{\| \alpha (f(x) - f(x_0) - Df(x_0)(x - x_0)) \| + \| \beta (g(x) - g(x_0) - Dg(x_0)(x - x_0)) \|}{\| x - x_0 \|}$$

The squeeze theorem gives

$$\lim_{x \to x_0} \frac{\| (\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0) - (\alpha Df(x_0) + \beta Dg(x_0))(x - x_0) \|}{\| x - x_0 \|} = 0$$

which says $D(\alpha f + \beta g)(x_0) = \alpha Df(x_0) + \beta Dg(x_0)$.

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4. If $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is a constant function, show that

$Df(x) = 0$ for all $x \in A$.

Proof. Note that $Df(x) = 0$ is an $m \times n$ matrix.
Let $x_0 \in A$.

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|}{\| x - x_0 \|} = \lim_{x \to x_0} \frac{\| K - K - 0 \cdot (x - x_0) \|}{\| x - x_0 \|}$$

$$= \lim_{x \to x_0} \frac{\| 0 \|}{\| x - x_0 \|} = \lim_{x \to x_0} 0 = 0.$$

So the derivative of $f(x) = K$ a constant is the $m \times n$ matrix of all zeros for every $x_0 \in A$.