2. Prove that \( f(x) = \frac{1}{x} \) is uniformly continuous on \([a, \infty)\) for \( a > 0 \).

Proof: Let \( \varepsilon > 0 \), then
\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right| < \frac{|y-x|}{a^2}.
\]
Let \( \delta = \frac{a^2 \varepsilon}{|y-x|} \). Then \( d(x,y) < \delta \) and \( x, y \geq a \) implies
\[
d(f(x), f(y)) = \left| f(x) - f(y) \right| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right| \leq \frac{|y-x|}{a^2} < \frac{a^2 \varepsilon}{a^2} = \varepsilon.
\]
Thus, \( f(x) = \frac{1}{x} \) is uniformly continuous on \([a, \infty)\).
4. If $f$ and $g$ are uniformly continuous maps of $\mathbb{R}$ to $\mathbb{R}$, must the product $f \cdot g$ be uniformly continuous?

What if $f$ and $g$ are bounded?

No. Example: Let $f(x) = g(x) = x$. Clearly these are uniformly continuous but in class we saw that $f \cdot g = x^2$ was not uniformly continuous.

Yes. Proof: Let $\epsilon > 0$. 

$$|f(x) \cdot g(x) - f(y) \cdot g(y)| = |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)|$$

$$\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)|$$

$$= |g(x)| |f(x) - f(y)| + |f(y)| |g(x) - g(y)|$$

Let $M$ be a bound for $|f(x)|$ and $N$ be a bound for $|g(x)|$. Since $f$ is uniformly continuous, $\exists \delta_f > 0$ such that $|x - y| < \delta_f \implies |f(x) - f(y)| < \frac{\epsilon}{2N}$, similarly since $g$ is uniformly continuous $\exists \delta_g > 0$ such that $|x - y| < \delta_g \implies |g(x) - g(y)| < \frac{\epsilon}{2M}$.

Now let $\delta = \min \{ \delta_f, \delta_g \}$ then
\[ d(x, y) < \delta \quad \text{and simpler} \quad d(f(x)g(x), f(y)g(y)) = \left| f(x)g(x) - f(y)g(y) \right| \]

\[ = \left| f(x)g(x) - f(y)g(y) + f(y)g(x) - f(y)g(y) \right| \]

\[ = \left| f(x)g(x) - f(y)g(y) \right| = \left| f(x) - f(y) \right| g(x) + \left| f(y) \right| g(x) - g(y) \]

\[ = \left| f(x) - f(y) \right| N + M \left| g(x) - g(y) \right| \]

\[ \leq \frac{\varepsilon}{2N} N + M \frac{\varepsilon}{2M} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
6. a. Show that \( f: \mathbb{R} \rightarrow \mathbb{R} \) is not uniformly continuous

iff there exist an \( \varepsilon > 0 \) and sequences \( x_n \) and \( y_n \)
such that \( |x_n - y_n| < \frac{1}{n} \) and \( |f(x_n) - f(y_n)| \geq \varepsilon \).

Generalize this statement to metric spaces.

b. Use a, on \( \mathbb{R} \) to prove that \( f(x) = x^2 \) is not
uniformly continuous.

Proof a. \( \Rightarrow \) Negating the definition of \( f \) is uniformly continuous

gives \( \exists \varepsilon > 0 \) for which there is no \( \delta > 0 \) such that
\( x, y \in \mathbb{R} \) and \( d(x, y) < \delta \) imply \( d(f(x), f(y)) < \varepsilon \). This means

that for this \( \varepsilon \) we can take \( \delta = \frac{1}{\varepsilon} \) and find points
\( x_n, y_n \in \mathbb{R} \) with \( \frac{1}{n} < d(x_n, y_n) < \frac{1}{n} \) and \( d(f(x_n), f(y_n)) \geq \varepsilon \).

Since we can do this for each \( n \), we have the desired sequence.

\( \Leftarrow \) the steps are logically reversible.

Generalization: Let \( f: A \subseteq M \rightarrow N \). It is not uniformly
continues iff there exist an \( \varepsilon > 0 \) and sequences \( x_n \) and \( y_n \)
in \( A \) such that \( d(x_n, y_n) < \frac{1}{n} \) and \( f(x_n), f(y_n) \geq \varepsilon \).
b. Let \( x_n = n \), \( y_n = n + \frac{1}{2n} \) and \( \epsilon = 1 \). Then

\[
|n - (n + \frac{1}{2n})| = \frac{1}{2n^2} < \frac{1}{n} \quad \text{and} \quad |n^2 - (n + \frac{1}{2n})^2| = \left| n^2 - n^2 - 1 - \frac{1}{4n^2} \right| = \frac{1}{4n^2} > \epsilon
\]
10. Show that \( f : A \rightarrow \mathbb{R}^n, A \subset \mathbb{R}^n \), is continuous iff for every set \( B \subset A \), \( f(\overline{B}) \cap A \subset \overline{f(B)} \).

**Proof:** \( \Rightarrow \) Let \( B \subset A \). If \( t \in B \) then \( f(t) \in f(B) \subset \overline{f(B)} \).

If \( t \in (\overline{B}) \setminus B \), then \( t \) is an accumulation point for \( B \). Thus there is a sequence of points \( t_k \) in \( B \) with \( t_k \to t \). By the continuity of \( f \), \( f(t_k) \to f(t) \).

As noted earlier \( f(t_k) \in f(B) \), thus \( f(t) \in \overline{f(B)} \) since a closed set contains all of its cluster points.

\( \Leftarrow \) I will show that the inverse image of a closed set is a relatively closed set. This is equivalent to continuity. Let \( C \) be a closed subset of \( \mathbb{R}^n \) and let \( B = f^{-1}(C) \). We are done if we show that \( \overline{B} \cap A = B \). Note that \( B \) is a subset of \( A \) and hence \( B \subset \overline{B} \cap A \). Let \( x \in \overline{B} \cap A \). By hypothesis \( f(\overline{B}) \cap A \subset \overline{f(B)} \). Hence \( f(x) \in \overline{f(B)} \). By the definition of \( B \), \( f(B) \subset C \).
Since $C$ is closed, $\text{Cl}(f(B)) \subseteq C$. Thus $f(x) \in \text{Cl}(f(B)) \subseteq C$, hence $B = f^{-1}(c)$, $x \in B$.

Since $x$ was an arbitrary point of $\text{Cl}(B) \cap A$, $\text{Cl}(B) \cap A \subseteq B$. So we have $\text{Cl}(B) \cap A = B$.

This completes the proof.
18. Let $A \subset \mathbb{R}$ be connected and let $f : A \to \mathbb{R}$ be continuous with $f(x) \neq 0$ for all $x \in A$, show that $f(x) > 0$ for all $x \in A$ or else $f(x) < 0$ for all $x \in A$.

Proof: Suppose there exist $x, y \in A$ with $f(x) < 0 < f(y)$. Since $A$ is connected and $f$ is continuous, $f$ satisfies the Intermediate Value Theorem. Thus there exists $z \in A$ with $f(z) = 0$. This contradicts $f$ is never zero, thus $f$ cannot have values on both sides of zero.
24.6. Let \( f: A \to M \to N \). Let \( f \) be uniformly continuous, and let \( x_k \) be a Cauchy sequence in \( A \). Show that \( f(x_k) \) is a Cauchy sequence.

Proof: Let \( \varepsilon > 0 \) then there exists \( \delta > 0 \) such that if \( d(x, y) < \delta \), \( x, y \in A \) then \( \rho(f(x), f(y)) < \varepsilon \). Since \( f \) is uniformly continuous, \( x_k \) is a Cauchy sequence in \( A \).

Let \( \varepsilon > 0 \) such that if \( n, m \geq 1 \) then \( d(x_n, x_m) < \delta \).

Since \( x_n, x_m \in A \), \( \rho(f(x_n), f(x_m)) < \varepsilon \). Thus \( f(x_k) \) is a Cauchy sequence.
28. Let \( f : (0, 1) \to \mathbb{R} \) be uniformly continuous. Must \( f \) be bounded?

Yes.

Proof: Let \( \varepsilon = 1 \). Then \( \exists \delta > 0 \) such that \( |x - y| < \delta \), \( x, y \in (0, 1) \), implies \( |f(x) - f(y)| < 1 \). I will show that

\[
|f(x)| \leq |f(\frac{x}{2})| + \frac{1}{\delta} + 1,
\]

since \( x \in (0, 1) \), \( |x - \frac{x}{2}| < \frac{\varepsilon}{2} \), \( \delta \) can be any number greater than 0.

Now assume \( x > \frac{1}{2} \),

\[
|f(x) - f\left(\frac{x}{2}\right)| = |f(x) - f\left(x - \frac{x}{2}\right) + f\left(x - \frac{x}{2}\right) - f\left(x - \frac{3x}{4}\right) + f\left(x - \frac{3x}{4}\right) - \cdots - f\left(x - \frac{x}{2}\right) | + f\left(x - \frac{x}{2}\right) - f\left(\frac{x}{2}\right)
\]

where \( x - \frac{n\delta}{2} > \frac{1}{2} \) but

\[x - \frac{(n+1)\delta}{2} \leq \frac{1}{2} \], note that this breaks the interval \((\frac{1}{2}, x)\) into intervals of length \( \frac{\delta}{2} \) or less.

\[
|f(x) - f\left(\frac{x}{2}\right)| \leq |f(x) - f\left(x - \frac{x}{2}\right)| + |f\left(x - \frac{x}{2}\right) - f\left(x - \frac{3x}{4}\right)| + \cdots + |f\left(x - \frac{3x}{4}\right) - f\left(\frac{x}{2}\right)|
\]

\[
\leq 1 + 1 + \cdots + 1 = n + 1,
\]

The number \( n \) of intervals satisfies \( x - \frac{(n+1)\delta}{2} \leq x - \frac{1}{2} > \frac{\delta}{2} \), \( 2x - 1 > n \delta \)

\[n < \frac{2x - 1}{\delta} < \frac{2 - 1}{\delta} = \frac{1}{\delta} \], thus \( |f(x) - f\left(\frac{x}{2}\right)| \leq n + 1 < \frac{1}{\delta} + 1 \) and

\[|f(x)| < |f\left(\frac{x}{2}\right)| + \frac{1}{\delta} + 1 \]. Thus \( f \) is bounded.
33. A set $A \subseteq \mathbb{R}^n$ is called relatively compact when $\overline{cl(A)}$ is compact. Prove that $A$ is relatively compact iff every sequence in $A$ has a subsequence that converges to a point in $\mathbb{R}^n$.

Proof: $\Rightarrow$ Assume that $A$ is relatively compact and that $x_k$ is a sequence in $A$. Then $x_k$ is a sequence in $\overline{cl(A)}$. Since $\overline{cl(A)}$ is compact, it is sequentially compact. Thus $x_k$ has a convergent subsequence. This subsequence converges to a point in $\overline{cl(A)} \subseteq \mathbb{R}^n$.

$\Leftarrow$ Let $x_k$ be a sequence in $\overline{cl(A)}$, each $x_k$ is a point in $A$ or an accumulation point of $A$. If $x_k \in A$, let $y_k = x_k$. If $x_k \notin A$, there exists a point $y_k \in A$ with $d(y_k, x_k) < \frac{1}{k}$. The sequence $y_k$ is a sequence in $A$ and hence has a convergent subsequence $y_{k_n}$. So there is $y_0 \in cl(A)$, $y_{k_n} \to y_0$.

Let $\varepsilon > 0$, there exists $N > 0$ such that $n \geq N$ implies $d(y_{k_n}, y_0) < \frac{\varepsilon}{2}$. Now choose $n \geq \max \left\{N, \frac{3}{\varepsilon} \right\}$. Then $d(x_{k_n}, y_0) \leq d(x_{k_n}, y_{k_n}) + d(y_{k_n}, y_0) < \frac{1}{2k_n} + \frac{\varepsilon}{2} = \varepsilon$. Thus $x_{k_n} \to y_0$. So $x_k$ has a convergent subsequence in $\overline{cl(A)}$. 

Thus $x_k \to y_0$, so $x_k$ has a convergent subsequence in $\overline{cl(A)}$. 

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Thus $\mathcal{L}(A)$ is sequentially compact and hence compact. This was our definition of $A$ being relatively compact.