MA 426 Test 1

MARCH 5, 2010

Dr. Franke   Name ____________________

1. (20%) Give definitions for each of the following making sure that you define each of the terms you use.

   (a) (M, d) is a metric space.

   (b) $A \subset M$ is open.

   (c) M is complete.

   (d) $A \subset M$ is connected.

2. (15%)Give a counter example for each statement.

   (a) $bd(cl(A)) = bd(A)$.

   (b) If N_k is a sequence of nonempty closed sets with $N_{k+1} \subset N_k$, then $\bigcap_{k=1}^{\infty} N_k \neq \emptyset$.

   (c) If A is connected then A is path connected.

3. (15%)

   (a) State the Heine-Borel property.

   (b) In $\Re^n$ what 2 conditions are equivalent to the Heine-Borel property.

   (c) Let $A = \{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots\}$. Find an open cover of A which has no finite subcover.

4. (15%)Use ε-δ methods to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y(x-y)}{x^2+y^2} = 0.$$

5. (15%)Let M be a compact metric space and $A \subset M$ be closed. Show that A is compact.

6. (20%) Let A' denote the set of accumulation points of the set A, where A is a subset of a metric space (M, d).

   (a) Prove that A' is a closed subset of M.

   (b) Is $(A')' = A'$ for all A? (Prove or give a counter example.)

1\. a. Let M be a nonempty set and $d : M \times M \to R \ni$

5

1. $d(x, y) \geq 0$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x, y, z \in M$

5 b. $A \subset M$ is open iff $\forall x \in A \ \exists \varepsilon > 0 \ \ni \ D(x, \varepsilon) \subseteq A$.

$D(x,\varepsilon) = \{ y \in M : d(x,y) < \varepsilon \}$

c. M is complete iff every Cauchy sequence in M converges

5 to a point in M.

5 d. $A \subset M$ is connected iff there does not exist to open sets U, V

such that $A \cap U \neq \phi$, $A \cap V \neq \phi$, $A \subseteq U \cup V$, $A \cap U \cap V = \phi$.

2\. a) $A = (-1, 0) \cup (0, 1)$ $\quad$ $bd\, A = \{-1, 0, 1\}$

5 $\quad cl\, A = [-1, 1]$ $\quad$ $bd\,(cl\, A) = \{-1, 1\}$

5 b. $N_k = [k, \infty)$

5 c. $\{ (x, \sin \frac{1}{x}) : x \in (-\infty, 0) \cup (0, \infty) \} \cup \{ (0, y) : y \in [-1, 1] \}$

3\. 5 a.) Every open cover has a finite subcover

5 b.) closed and bounded

5 c.) $\left(\frac{3}{2}, 3\right), \left(\frac{4}{3}, 3\right), \ldots, \left(\frac{n+2}{n+1}, 3\right) \ldots$

4\.

$$\left|\frac{x^2 y (x-y)}{x^2+y^2}\right| \le \boxed{\frac{x^2 \sqrt{x^2+y^2}\,(2\sqrt{x^2+y^2})}{x^2+y^2}}^{\,3} = 2x^2 < \boxed{2(x^2+y^2)}^{\,2}$$

$\boxed{\text{Let } \varepsilon > 0}$ 2 let $\boxed{\delta = \frac{\sqrt{\varepsilon}}{2}}$ 2. Then $\underbrace{0 < \|(x,y)\| < \delta}_{2} \Rightarrow \underbrace{0 < \sqrt{x^2+y^2} < \delta}_{1}$

$$\left|\frac{x^2 y (x-y)}{x^2+y^2}\right| \le 2(x^2+y^2) < 2\cdot\delta^2 = 2\cdot\frac{\varepsilon}{4} = \underbrace{\frac{\varepsilon}{2} < \varepsilon}_{2}$$

1

5\. Since A is closed $M \setminus A$ is open, call it W. Let $\mathcal{U} = \{U_i\}$ be an open cover of A. Then $\mathcal{W} = \{W\} \cup \{U_i\}$ is an open cover of M. Since M is compact, a finite number of these cover M. Say $\{W, U_1, U_2, \ldots, U_n\}$.

Now since $A \cap W = \emptyset$, $A \subset \bigcup_{i=1}^{n} U_i$.

Thus $\mathcal{U}$ has a finite subcover of A.

Thus A is compact.

6.

a. Let x be an accumulation point of A'.
3
~~Thus~~ Let U be a neighborhood of x. Then U contains a point
2
$y \neq x$ and $y \in A'$. ~~Since~~ $U \cap D(y, d(x,y))$ is a neighborhood of y, $\exists z \in A$
2 2 2
such that $z \in U \cap D(y, d(x,y))$. Thus every neighborhood of x contains
a point of A not equal to x. Thus $x \in A'$.
2 1

1 Thus A' contains all of its accumulation points.
Thus A' is closed.

6. a. ~~A~~ $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$.
5

$A' = \{0\}$

$(A')' = \emptyset$.