MA 425 Test 1
SEPTEMBER 17, 2010

Dr. Franke   Name ______________________

1. (24%) Give definitions for each of the following making sure that you define each of the terms you use.

   (a) the supremum of a set S.

   (b) the completeness axiom.

   (c) (x_n) is a convergent sequence.

   (d) the Archimedean property

2. (15%) Prove using the definition of convergence that

$$\lim \left(\frac{5n-4}{2n+7}\right) = \frac{5}{2}.$$

3. (16%) True or False

   (a) _____ If $f : A \to B$ is injective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is injective.

   (b) _____ If $a, b \in \mathbb{R}$, and $a \neq b$, then there exists an $\varepsilon > 0$ such that $V_\varepsilon(a) \cap V_\varepsilon(b)$ is empty.

   (c) _____ Let S be a nonempty subset of $\mathbb{R}$ and let f and g be functions from S into D, a bounded subset of $\mathbb{R}$. Then $\sup\{f(x) + g(x) : x \in S\} = \sup\{f(x) : x \in S\} + \sup\{g(x) : x \in S\}$.

   (d) _____ If $a, b \in \mathbb{R}$ with $a < b$, then there is an irrational number q with $a < q < b$.

4. (21%) Let $X = (x_n)$ be a bounded sequence and $Y = (y_n)$ be a sequence such that $\lim(y_n) = 0$.

   (a) Define $X = (x_n)$ is a bounded sequence.

   (b) Prove that $\lim(x_n y_n) = 0$.

5. (24%)Suppose S and U are nonempty sets such that each element of U is an upper bound for S.

   (a) Prove that each element in S is a lower bound for U.

   (b) Prove $\sup(S) \leq \inf(U)$.

1 a. i. Let S be a set. a is an upper bound for S iff $s \le a \ \forall \ s \in S$.

ii. L is the supremum of a set S iff $L \le a$ for every upper bound a of S and L is an upper bound of S.

b. Every nonempty bounded set has a supremum.

c. Let (x_n) be a sequence. A sequence is a function from $\mathbb{N}$ into $\mathbb{R}$. (x_n) converges iff $\exists \ L \in \mathbb{R} \ni \forall \ \varepsilon > 0 \ \exists \ K \in \mathbb{N} \ni n \ge K, n \in \mathbb{N}$ $\Rightarrow |x_n - L| < \varepsilon$.

d. ~~Every~~ If $x \in \mathbb{R}$ then $\exists \ n \in \mathbb{N} \ni x \le n$.

2. Scratch work $\left|\frac{5n-4}{2n+7}-\frac{5}{2}\right| = \left|\frac{10n-8-10n}{4n+14}\right|$

$$= \frac{8}{4n+14} < \frac{8}{4n} = \frac{2}{n} < \varepsilon$$

$$\frac{2}{\varepsilon} < n$$

Proof: Let $\varepsilon > 0$ then $\frac{2}{\varepsilon} > 0$. By archimedean Property

$\exists\ K \in \mathbb{N} \ni \frac{2}{\varepsilon} < K$. If $n \geq K$ and $n \in \mathbb{N}$ then

$$\left|\frac{5n-4}{2n+7}-\frac{5}{2}\right| = \left|\frac{10n-8-10n}{4n+14}\right| = \frac{8}{4n+14} < \frac{8}{4n} = \frac{2}{n}$$

$$\leq \frac{2}{K} < \frac{2}{2/\varepsilon} = \varepsilon.$$

Thus $\lim\left(\frac{5n-4}{2n+7}\right) = \frac{5}{2}$.

3. a. False
b. True
c. False
d. True

4. a.) $X = (x_n)$ is bounded iff $\exists M \ni |x_n| \leq M \quad \forall n \in \mathbb{N}$.

b.) Since (x_n) is bounded $\exists M \ni |x_n| \leq M \quad \forall n \in \mathbb{N}$.

~~Then~~ Let $\bar{M} = M+1$. Then $|x_n| \leq \bar{M} \quad \forall n \in \mathbb{N}$ and $\bar{M} > 0$.

Let $\varepsilon > 0$ then $\frac{\varepsilon}{\bar{M}} > 0$. By Archimedean Property $\exists K \in \mathbb{N}$

$\ni$ if $n \geq K$ and $n \in \mathbb{N}$ then $|y_n| < \frac{\varepsilon}{\bar{M}}$.

Thus $|x_n y_n| = |x_n||y_n| \leq \bar{M} \cdot |y_n| < \bar{M} \frac{\varepsilon}{\bar{M}} = \varepsilon$.

Thus $\lim (x_n y_n) = 0$.

5.

a.) Let $a \in S$. Let $b \in U$. Since each element of U is an upper bound of S. $a \leq b$. Since b is an arbitrary element of U. $a \leq b$ $\forall$ $b \in U$. Thus a is a lower bound for U. Since a is an arbitrary element of S, every element of S is a lower bound for U.

b.) Since S and U are nonempty sets which are bounded above and below respectively, $\sup S$ and $\inf U$ are real numbers by completeness.

Suppose $\inf U < \sup S$. Then $\inf U$ is not an upper bound for S. Thus $\exists$ $a \in S$ $\ni$ $\inf U < a$.

Thus a is not a lower ~~bound~~ bound for U. $\Rightarrow \Leftarrow$ every element of S is a lower bound for U.

Therefore ~~$\inf U \leq$~~. $\sup(S) \leq \inf(U)$.