6.2

5. Show that \( f'(x) < 0 \) for \( x > 1 \). Then \( f \) is strictly decreasing on \([1, \infty)\) so that \( f(a/b) < f(1) \) for \( a > b > 0 \).

8. If \( \delta > 0 \) and \( a+h < b \), there exists \( c \in (a, a+h) \) such that \( f(a+h) - f(a) = h f'(c) \). Since \( c \to a \) as \( h \to 0^+ \), it follows that \( f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \) and \( f'(c_h) = a \). Now consider \( h < 0 \).


Page 201

7.1

4. (b) If \( u \in U_2 \), then \( u \in [x_{i-1}, x_i] \) with tag \( t_i \in [1, 2] \), so that (i) \( x_{i-1} \leq t_i \leq 2 \) which implies that \( u \leq x_i \leq x_{i-1} + ||P|| \leq 2 + ||P|| \) and (ii) \( 1 \leq t_i \leq x_i \) which implies that \( 1 - ||P|| \leq x_i - ||P|| \leq x_{i-1} \leq u \). Therefore \( u \) belongs to \([1 - ||P||, 2 + ||P||]\).

On the other hand, if \( 1 + ||P|| \leq v \leq 2 - ||P|| \) and \( v \in [x_{i-1}, x_i] \), then (i) \( 1 + ||P|| \leq x_i \) which implies that \( 1 \leq x_i - ||P|| \leq x_{i-1} \leq t_i \) and (ii) \( x_{i-1} \leq 2 - ||P|| \) which implies that \( t_i \leq x_i \leq x_{i-1} + ||P|| \leq 2 \). Therefore we get \( t_i \in [1, 2] \).

6. (a) If \( P \) is a tagged partition of \([0, 2]\), let \( \hat{P}_1 \) be the subset of \( \hat{P} \) having tags in \([0, 1]\), and let \( \hat{P}_2 \) be the subset of \( \hat{P} \) having tags in \([1, 2]\). The union of the subintervals in \( \hat{P}_1 \) contains the interval \([0, 1 - ||\hat{P}||]\) and is contained in \([0, 1 + ||\hat{P}||]\), so that \( 2(1 - ||\hat{P}||) \leq S(f; \hat{P}_1) \leq 2(1 + ||\hat{P}||) \). Similarly, the union of the subintervals in \( \hat{P}_2 \) contains \([1 + ||\hat{P}||, 2]\) and is contained in \([1 - ||\hat{P}||, 2]\), so that \( 1 - ||\hat{P}|| \leq S(f; \hat{P}_2) \leq 1 + ||\hat{P}|| \). Therefore \( 3 - 3||\hat{P}|| \leq S(f; \hat{P}) = S(f; \hat{P}_1) + S(f; \hat{P}_2) \leq 3 + 3||\hat{P}|| \), whence \( S(f; \hat{P}) - 3 \leq 3||\hat{P}|| \), and we should take \( ||\hat{P}|| < \varepsilon/3 \).

8. Since \( -M \leq f(x) \leq M \) for \( x \in [a, b] \), Theorem 7.1.4(c) implies that we have \( -M(b - a) \leq \int_a^b f \leq M(b - a) \) whence the inequality follows.

9. Given \( \varepsilon > 0 \) there exists \( \delta_\varepsilon > 0 \) such that if \( ||\hat{P}|| < \delta_\varepsilon \) then \( |S(f; \hat{P}) - \int_a^b f| < \varepsilon \). Since \( ||\hat{P}_n|| \to 0 \), there exists \( K_\varepsilon \) such that if \( n > K_\varepsilon \) then \( ||\hat{P}_n|| < \delta_\varepsilon \), whence \( |S(f; \hat{P}_n) - \int_a^b f| < \varepsilon \). Therefore, \( \int_a^b f = \lim_n S(f; \hat{P}_n) \).

11. If \( f \in \mathcal{R}[a, b] \), then Exercise 9 implies that both sequences of Riemann sums converge to \( \int_a^b f \).

12. Let \( P_n \) be the partition of \([0, 1]\) into \( n \) equal parts. If \( \hat{P}_n \) is this partition with rational tags, then \( S(f; \hat{P}_n) = 1 \), while if \( Q_n \) is this partition with irrational tags, then \( S(f; Q_n) = 0. \)