1. \( y_8 = 3 \cdot 2^4 \)
\[ 56 = 7 \cdot 2^3 \]

56 is to the left of 7, in the Sankowskii ordering, so there is a continuous map of \( \mathbb{R} \) with period 7, but not one of period 56.

2. \( 176 = 11 \cdot 2^4 \)
\[ 96 = 3 \cdot 2^5 \]

176 is to the left of 96 in the Sankowskii ordering, so if a continuous map of \( \mathbb{R} \) has a point of prime period 176, it must also have one of period 96.

3. It cannot happen if \( F \) is continuous. But an example where \( F \) is not continuous is

\[ F(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in \left[ \frac{1}{2}, \frac{3}{4} \right] \\ 0 & \text{if } x \in \left( \frac{3}{4}, 1 \right] \end{cases} \]

\[ F(0) = \frac{1}{2}, \quad F(\frac{1}{2}) = 1, \quad F(1) = 0 \]
4. in the first graph, \( F([0,2]) = [1,3] \supset [1,3] \)
and \( F([3,5]) = [0,3] \) the entire interval.
This is very much like the conditions in point 3
implies above.

Let \( A_1 \) be a closed interval in \([1,3]\) with
\( F(A_1) = [1,3] \). Let \( A_2 \subset [0,3] \) be a closed interval
with \( F(A_2) = A_1 \), there is such a \( A_2 \) with \( F([1,3]) \supset A_1 \).

Since \( F([1,3]) \supset A_2 \), \( A_2 \) are closed intervals in \([3,5]\)
with \( F(A_2) = A_2 \).

Now \( F^2(A_2) = F(A_2) = F(A_1) = [1,3] \supset A_3 \).

So there is a fixed point \( x^* \) for \( F^3 \) in \( A_3 \). Now
\( F(x^*) \in A_2 \subset [0,3] \). \( x^* + 2 \) minus 2 has prime
period 4. Thus \( x^* \) is not fixed. Its orbit is in
\([1,3]\) for two iterations and then in \([3,5]\). Thus
it has prime period 3.

By Sarkovski's ordering it has points of every
prime period.
F(0, 1) = [0, 1], F(E9, 7] = [1, 5], F(C5, 3] = [3, 7]
F([3, 7]) = [1, 6], F(E5, 6] = [5, 7].

{1, 5} \cap \mathbb{F}_5([1, 5]) = 2, a point of period 7 out 5.

Similarly:
{2, 7} \cap \mathbb{F}_5([2, 7]) = [2, 7] \cap [3, 7] = 3
{3, 4} \cap \mathbb{F}_5([3, 4]) = [3, 4] \cap [4, 7] = 4
{4, 6} \cap \mathbb{F}_5([4, 6]) = [4, 6] \cap [5, 6] = 5
{5, 7} \cap \mathbb{F}_5([5, 7]) = [5, 7] \cap [5, 6] = 6.

These intersections are all period 7 out period 5.
So the only interval left is [E9, 5].
F([E9, 5]) = [E3, 5] decreasing
F([E3, 5]) = [E3, 6] decreasing
F([E3, 6]) = [E3, 6] decreasing
F([E3, 6]) = [E3, 6] decreasing
F([E3, 7]) = [E3, 7] decreasing
F([E3, 7]) = [E3, 7] decreasing

F is decreasing on [E9, E5] as it has only one fixed point which is the original fixed point, the previous period 5.
7. The iterate values give a point of period 6.

\[ F([4, 8]) = [4, 6] \]
\[ F([4, 6]) = [1, 3] \]

Points in these two intervals go back and forth. No point in them can have odd period.

\[ F([4, 5]) = [1, 4] \] and it is decreasing. The points that get out must stay out, see the first argument.

For odd iterations of [3, 4] the maps decreasing on what remains in [3, 4]. So it has been only one fixed point. But this is the original fixed point of period 1. There no point has prime period an odd iterate larger than 1.
1. Give definitions for each of the following making sure that you define each of the terms you use.

   (a) (6%) \( f : [a, b] \to \mathbb{R} \) is strictly increasing.
   (b) (6%) \( A \subseteq \mathbb{R} \) is a null set.

2. (12%) State our Fundamental Theorem of Calculus: both forms.

3. (6%) State the Intermediate Value Theorem for Derivatives (Darboux's Theorem).

4. (35%) Let

   \[ g(x) = \begin{cases} 
   x + 2x^2 \sin(1/x) & x \neq 0 \\
   0 & x = 0 
   \end{cases} \]

   (a) Show that \( g'(0) = 1 \).
   (b) Is \( g(x) \) continuous at 0? (explain)
   (c) Is \( g'(x) \) continuous at 0? (explain, Hint: \( x_n = \frac{1}{n\pi} \))
   (d) Is there a \( \delta > 0 \) such that \( g \) is increasing on \( (-\delta, \delta) \)? (Why or why not?)

5. (20%) True or False

   (a) _______ If \( f : [a, b] \to \mathbb{R} \) is strictly increasing and is differentiable at \( x_0 \in (a, b) \), then \( f'(x_0) > 0 \).
   (b) _______ If \( f \) has a continuous derivative on \( [a, b] \), then \( f \) satisfies a Lipschitz condition on \( [a, b] \).
   (c) _______ If \( f \) is bounded on \( [a, b] \), then it is Riemann integrable.
   (d) _______ If \( f \) is continuous on \( [a, b] \), then it is Riemann integrable.
   (e) _______ If \( f : [a, b] \to \mathbb{R} \) is Riemann integrable on \( [c, b] \) for every \( c \in (a, b) \), then \( f \) is Riemann integrable on \( [a, b] \).

6. (15%) Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable and \( c \in \mathbb{R} \). Define \( g : [a+c, b+c] \to \mathbb{R} \) by \( g(x) = f(x-c) \). Prove that \( g \) is Riemann integrable and that

   \[ \int_{a+c}^{b+c} f - \int_c^{b-c} g. \]

   (Hint: Compare the Riemann sums for \( f \) and \( g \).)
a. \( g'(x) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x + 2 \sin \frac{1}{x} - 0}{x - 0} \)

\[= \lim_{x \to 0} 1 + 2 \cos \frac{1}{x} = 1 + 0 = 1 \]

b. Differentiability implies continuity, so \( g \) is continuous at \( 0 \).

c. \( g'(x) = \begin{cases} 
1 & x > 0 \\
1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x} & x < 0
\end{cases} \)

\( g'(\frac{1}{2\pi}) = 1 + \frac{4}{2\pi} \sin(\frac{1}{2\pi}) - 2 \cos(\frac{1}{2\pi}) = 1 - 2 = -1 \)

\( \lim g'(\frac{1}{2\pi}) = -1 \neq g'(0) \) and \( \lim g'(\frac{1}{2\pi}) = 0 \),

so \( g' \) is not continuous at \( 0 \).

d. \( g' \) is continuous at all other points except \( 0 \).

There is no \( b > 0 \) such that \( g \) is increasing on \((-b, b)\).

Let \( b > 0 \) and \( \frac{1}{2\pi} < b \). \( g'(\frac{1}{2\pi}) = -1 \). \( g' \) will be negative on an interval inside \((-b, b)\). \( g \) will be decreasing on this sub-interval. \( g \) is not decreasing on \((-\infty, b)\).
5. a. F
   b. T
   c. F
   d. T
   e. F

6. Let $\epsilon > 0$, given $f \in R[a,b]$, $\exists \delta > 0$ if $||P|| < \delta$ then $
\left| S(f, P) \right| < \epsilon$. Let $Q$ be a tagged partition of $[a+\delta, b+\delta]$, with $||Q|| < \delta$. Take every point in the partition $Q$ of $[a+\delta, b+\delta]$ and subtract $c$. Then these points give a partition $P$ of $[a, b+\delta]$. If we add any point of $Q$ as denoted by $c$, we get $S(f, P)$. Then

$$||Q|| = ||P|| + \delta$$

and

$$S(f, Q) = S(f, P).$$

Thus

$$\left| S(f, Q) - \int_a^b f \right| = \left| S(f, P) - \int_a^b f \right| < \epsilon$$

Thus $g \in R[a+\delta, b+\delta]$ and

$$\int_{a+\delta}^{b+\delta} g = \int_a^b f.$$