MA 225 Test 2

OCTOBER 14, 2005

Dr. Franke

Name ____________________

Show all work. You may not use a calculator.

1. (16%) State the following definitions and/or theorems:

   (a) The Well Ordering Principle

   (b) $\bigcap_{\alpha \in \Delta} A_\alpha$ where $\{A_\alpha | \alpha \in \Delta\}$ is an indexed collection of sets.

2. (16%) Prove the following statement using mathematical induction

$$3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n \quad \forall n \in \mathbb{N}.$$

3. (16%) Determine the natural numbers n for which $n^2 > 2n + 3$ and prove your result using mathematical induction.

4. (20%) For each of the following, determine if it is true or false.

   (a) $\{a\} \subseteq \mathcal{P}(\{a, \varnothing\})$

   (b) $\{\varnothing\} \subseteq \mathcal{P}(\{a, \varnothing\})$

   (c) $\{\varnothing\} \in \mathcal{P}(\{a, \varnothing\})$

   (d) $\bigcap_{n=1}^{\infty} [\frac{1}{n}, 2 + \frac{1}{n}) = [1, 2)$.

5. (16%) Find counter examples to show that each of these is false.

   (a) Let A, B and C be sets. Then $A - (B - C) = (A - B) - C$.

   (b) If $A \subseteq B \cup C, B \subseteq A \cup C, C \subseteq A \cup B$ and $A \cap B = A \cap C$, then $A = B$.

6. (16%) Prove that if A, B, C and D are sets and $A \subseteq B$ and $C \subseteq D$, then $C - B \subseteq D - A$.

1. a. Every nonempty subset of $\mathbb{N}$ has a smallest element.

b. $\bigcap_{\alpha \in \Delta} A_\alpha = \{x : \forall \alpha \in \Delta,\ x \in A_\alpha\}$

2. Let $P(n)$ be $3+11+19+\cdots+(8n-5) = 4n^2-n$.

a. $P(1)$ is $3 = 4\cdot 1^2 - 1 = 3$ which is true.

b. Assume $P(n)$ is true for some fixed $n \in \mathbb{N}$.

$P(n+1)$ is $3+11+19+\cdots+(8n-5)+(8(n+1)-5) = 4(n+1)^2-(n+1)$.

This what we want to get.

Now $3+11+19+\cdots+(8n-5)+(8(n+1)-5) = 4n^2-n+(8(n+1)-5)$

$= 4n^2-n+8n+8-5 = 4n^2+7n+3 = 4n^2+8n+4-n-1$

$= 4(n^2+2n+1)-(n+1) = 4(n+1)^2-(n+1)$.

Thus $P(n)$ implies $P(n+1)$.

Hence by PMI, $P(n)$ is true for every $n \in \mathbb{N}$.

3. $\phi^2 < 2\cdot 1+3 = 5$

$4 = 2^2 < 2\cdot 2+3 = 7$

$9 = 3^2 = 2\cdot 3+3 = 9$

$16 = 4^2 > 2\cdot 4+3 = 11$

So $n^2 > 2n+3$ is first true when $n = 4$.

Assume $n^2 > 2n+3$ for some fixed $n \geq 4$.

$(n+1)^2 = n^2 + 2n + 1 > 2n+3 + 2n+1 \geq 2n+3+2\cdot 1+1 \geq 2n+2+3$

$= 2(n+1)+3.$

Thus $(n+1)^2 > 2(n+1)+3$.

Hence by generalized PMI, $n^2 > 2n+3$ is true for all $n \geq 4$.

4. a.) F
   b) T
   c.) T
   d.) F

5. a. $A = \{1, 2, 3\}$ $A - (B - C) = \{1, 2, 3\} - \{2\} = \{1, 3\}$

$B = \{2, 4\}$ $(A - B) - C = \{1, 3\} - \{3, 4\} = \{1\}$

$C = \{3, 4\}$

b. $A = \{1\}$

$B = \{1, 2\}$

$C = \{1, 2\}$

6. Let A, B, C and D be sets with $A \subseteq B$ and $C \subseteq D$.

Let $x \in C - B$. Then $x \in C$ and $x \notin B$.

Then since $C \subseteq D$, $x \in D$. And since $A \subseteq B$, $x \notin A$. Thus $x \in D$ and $x \notin A$. Thus $x \in D - A$. Since x was arbitrary.

$C - B \subseteq D - A$.