Show all work. You may not use a calculator.

1. (15%) Define the following terms:
   (a) \( f \) is a function from \( A \) to \( B \).
   (b) \( f : A \rightarrow B \) is one to one.
   (c) \( f : A \rightarrow B \) is onto.

2. (18%) Show each of the following statements is false by finding a counterexample. In each case, assume \( f \) and \( g \) are functions. (Remember: a counterexample for \( P \Rightarrow Q \) is an example in which \( P \) is true and \( Q \) is false.
   (a) If \( f : A \rightarrow B \) then \( f^{-1} \) is a function from \( B \) to \( A \).
   (b) If \( f : A \rightarrow B \) and \( g : C \rightarrow D \), then \( f \cup g \) is a function with range \( B \cup D \).
   (c) If \( f : A \rightarrow B \) and \( g : B \rightarrow C \) and \( g \circ f : A \rightarrow C \), then \( f : A \rightarrow B \).

3. (27%) In the following problem, let \( S = \{ c \mid c \) is a circle in the plane\( \) and \( T = \{ c \mid c \) is a circle in the plane and \( c \) is centered at the origin\( \}. \) For each of the following relations, determine if it is a function. If it is a function, determine if it is 1-1. Also determine if it is onto its codomain. A single point is not a circle. Justify your answers.
   (a) \( R_1 = \{(c, r) \in S \times (0, \infty) \mid r \) is the radius of \( c \}\)
   (b) \( R_2 = \{(r, c) \in (0, \infty) \times S \mid r \) is the radius of \( c \}\)
   (c) \( R_3 = \{(r, c) \in (0, \infty) \times T \mid r \) is the radius of \( c \}\)

4. (10%) Find the domain of the function \( f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = \frac{x}{x+1}\}. \) Is this function 1-1? Justify your answer.

5. (15%) Prove: If \( f \) and \( g \) are functions, \( f : A \rightarrow B \) and \( g : B \rightarrow C \), then \( g \circ f : A \rightarrow C \).
   (Note: you do not need to prove \( g \circ f \) is a function. You just need to prove it is one to one.)

6. (15%) Note: \( f \) is an increasing function on \( A \subseteq \mathbb{R} \), if \( f : A \rightarrow \mathbb{R} \) and \( \forall x, y \in A, x < y \implies f(x) < f(y) \). Prove: If \( f : A \rightarrow \mathbb{R} \) is an increasing function, then \( f^{-1} : \{y \mid f(y) \in A \} \rightarrow A \) is an increasing function.
1. a) A function from A to B is a relation from A to B such that
i. dom f = A
ii. If (x, y) and (x, y) ∈ f then y ≠ z.
b) f: A → B is one-to-one if f(x) = f(y) then x = y.
c) f: A → B is onto if range f = B.

2. a) A = \{0, 1\} B = \{2\}
f = \{(0, 2), (1, 2)\}
f⁻¹ = \{(2, 0), (2, 1)\}
f⁻¹ is not a function.
b) A = \{0, 1\} B = \{2\}
c = \{1, 3\} D = \{2, 4\}
f = \{(0, 2), (1, 2)\}
g = \{(1, 4), (3, 2)\}
f ∘ g = \{(0, 2), (1, 2), (1, 4), (3, 2)\} and a function.
c) A = \{1\} B = \{2, 3\} C = \{5\}
f = \{(1, 2)\} not with B
f = \{(2, 5), (3, 5)\}
g f = \{(1, 5)\} not with C.
3. a) \( h \) is a function that is onto but not 1-1.

   Many circles circling plane have the same radius.

b) \( h \) is not a function because given a positive number it matches many circles radii.

c) \( h \) is a function that is 1-1 and onto. There is exactly one circle centered at \((0,0)\) with a fixed radius.

4. \( \operatorname{Dom} f = \{ x \in \mathbb{R} : x \neq \pm \frac{1}{2} \} \)

   Suppose \( \frac{x+1}{x-1} = \frac{y+1}{y-1} \)

c) \((x+1)(y-1) = (x-1)(y+1)\)

   d) \(x^2 + 2x - 1 = x^2 - 2x + x - 1\)

   e) \(2x = -2 + 1\)

   f) \(x = -\frac{1}{2}\).

   Thus \(f \circ h = 1-1\).

5. Assume \((x, z) \text{ and } (y, z) \in \text{Dom} f \).

   Then

   \( g(f(x)) = g(f(y)) \) or \( g(f(\theta)) = g(f(\phi)) \). Since \( g \circ f = 1-1 \),

   \( f(\theta) = \phi \). Then since \( f \circ h = 1-1, x \neq y \). Thus

   \( g \circ f \circ h = 1-1 \).
6. To see that $f^{-1}$ is a function, let

$$(x, z) \text{ and } (y, z) \in f^{-1}.$$ Then $(z, x) \in f$. If $z \preceq y$ then $f(z) \preceq f(y)$ and $x \preceq x \preceq y$. Thus $f^{-1}$ is a function. Since $f^{-1} \cap \text{rng } f = \emptyset$.

Let $x, y \in \text{dom } f^{-1}$ with $(x, z) \in f^{-1}$ and $(y, z) \in f^{-1}$. Then $(z, x) \in f$ and $(z, y) \in f$.

If $s \preceq t$ then $f(s) > f(t)$ and $x \preceq x \preceq y$. If $s \preceq t$ then $f(s) = f(t)$ and $x \preceq x \preceq y$.

Thus $x \preceq y$ with $x, y \in \text{dom } f^{-1}$.

$$f^{-1}(x) < f^{-1}(y).$$