Math 225

WRITING MATHEMATICS

In any discipline, the successful communication of ideas is at least as important as the ideas themselves. Most disciplines develop standard usages and restrictions that differ from everyday English. Mathematics is not an exception. This handout is intended to alert you to some (but certainly not all) of the peculiarities of writing mathematics.

THE PURPOSE OF WRITTEN ASSIGNMENTS

In Math 225, the purpose of written homework is to develop the skills which are essential for understanding and communicating mathematics. Your homework paper is not a certificate proving that you have done the assignment. Rather, it is an exercise in both learning and in reporting what you have learned. An important way of deepening your understanding of mathematics is to explain it to another person.

YOUR RESPONSIBILITIES

Your first responsibility is to communicate with the reader. Do not write to the instructor (- I already know how to do the problem). Instead, assume your reader is someone who needs your help. A good guideline for this course is to include details to the extent that, should you find your work again in five years, you could understand it fully on the first reading. Make sure your write-up is clear, logical, and complete. Your arguments should lead the reader from one step to the next without any leaps of faith or verbal contortions.

A crucial test of your work is this: *can another person learn from your paper? easily?* Remember that the reader will only see what you wrote and will not know what you meant to say. So all of the necessary details must be there, and, of course, your paper must be accurate.

Remember that you are writing to an audience. So, make sure that your paper is *"reader-friendly."*

GROUND RULES

There are two components of writing mathematics: mathematics (content) and writing (form). Each is important and neither can be ignored.

• Every statement you make must be mathematically correct. If you are not sure of the validity of a claim, first convince yourself. Then include in your write-up the details you required in order to reach your conclusion. It is better to make no assertion than to make a false assertion.

Example. For every natural number n, $n^2 \leq 2^n$. Hence $2n^2 \leq 2^{n+1}$ for every natural number n.

The first statement is false. While $n^2 \leq 2^n$ for *some* natural numbers n, there are natural numbers for which the inequality, $n^2 \leq 2^n$, *doesn't* hold. For example, $3^2 = 9$, while $2^3 = 8$. Hence for n = 3, $n^2 \leq 2^n$ is false. So the first sentence is wrong and thus the author cannot (yet) conclude that $2n^2 \leq 2^{n+1}$ for every natural number n. (Actually, the second assertion is false as well since $2(3^2) = 18 > 16 = 2^{3+1}$.)

MI **Example.** For every natural number n, 5^n is odd.

This statement is certainly not obvious. The author should include details to justify the claim. For example, it would be appropriate to supply the following induction proof of this statement:

> Let $S = \{n \in \mathbf{N}: 5^n \text{ is odd}\}$. Since $5^1 = 2(2) + 1$, 5^1 is odd and thus $1 \in S$. Assume that $n \in S$. As 5^n is odd, there exists an integer m such that $5^n = 2m + 1$. Then $5^{n+1} = (5^n)5 = (2m + 1)5 = 10m + 5 = 2(5m + 2) + 1$. Since m is an integer, so is 5m + 2. Therefore 5^{n+1} is odd as well. Thus $n + 1 \in S$ whenever $n \in S$. So, by the Principle of Mathematical Induction, $S = \mathbf{N}$.

MI **Example.** For every natural number k, $k^4 + k^2 + 37$ is prime.

This statement is also far from obvious. In fact, it is incorrect. When k = 37,

$$k^4 + k^2 + 37 = (37)^4 + (37)^2 + 37 = 37\,(37^3 + 37 + 1),$$

which has factors 37 and $(37^3 + 38)$.

- The rules of standard grammar remain in effect when writing mathematics. In particular, *you should use complete sentences and well-structured paragraphs.*

Example. If a is an integer.

What happens if a is an integer? The author has left the reader in suspense with this incomplete sentence.

Example. Suppose a =.

This is not a complete sentence. The verb "=" requires an object.

Example. $(a + b)^2 = a^2 + 2ab + b^2$.

This is a perfectly valid complete sentence. It has subject "$(a + b)^2$", verb "=" and object "$a^2 + 2ab + b^2$." Many other mathematical symbols may also serve as verbs (e.g., $>$, $\geq$, $\exists$, $\in$, $\subseteq$, $\Rightarrow$, etc.).

Example. Suppose $\frac{b}{a}$. Then b = ac for some integer c.

No verb appears in the first sentence. The author probably intended to say, "Suppose a divides b." The symbol for "a divides b" is "$a \mid b$."

RESTRICTIONS ON WORD AND SYMBOL USAGE

In everyday English, symbols are used infrequently. In mathematics, they are used regularly as an integral facet of communication. As a result, mathematics places additional restrictions on the use of certain symbols.

- Symbols that have a specific mathematical meaning are reserved for mathematical use.

  Example. Let $a + b$ be positive. Then the product ab is also positive.

  While the author may have intended for a **and** b each to be greater than zero, he or she has instead indicated that the quantity $a + b$ is greater than zero. The misstatement allows, among others, the case $a = 0$ and $b = 1$. Indeed, in that case, $a + b = 1$, a positive number, but the product ab is 0, which is not positive. The author should replace the symbol "+" by the written word "and."

- Always respect the "equals" sign.

  Example. Let n = even.

  The variable name "n" is not equal to the word "even." The author should instead say, "Let n *be* even."

  Example. Case 1 = n is odd.

  The author does not mean that the phrase "Case 1" is *equal* to the phrase "n is odd." One acceptable alternative would be to use a colon in place of the equals symbol.

  Example. $(P \Rightarrow Q) = (Q \vee \sim P)$.

  While the statements on either side of the equals sign have the same truth table, and hence are *equivalent* statements, they are *not* equal. The author should instead say:

  $$P \Rightarrow Q \text{ is equivalent to } Q \vee \sim P$$

  or

  $$P \Rightarrow Q \text{ iff } Q \vee \sim P.$$

• Mathematics is case-sensitive.

Example. If $a = 3$, then $A + 4 = 7$.

The author here has changed a to A. While a has been defined, A has not and so the implication does not necessarily hold. Do not use upper-case and lower-case versions of the same letter interchangeably unless you specifically define them to represent the same quantity.

• More generally, define any terms you use.

Example. For any polygon, the sum of the interior angles is $(n - 2)180$.

The author has never told the reader what n is and so the statement conveys no practical information. It would be preferable to say, for example:

> For any polygon, the sum of the interior angles is $(n - 2)180$, where n is the number of sides in the polygon.

• Once you've assigned a variable name, don't re-use it for a different meaning within the same context.

Example. Suppose m and n are two even numbers. Then each is twice an integer, that is, $m = 2b$ for some integer b and $n = 2b$ for some integer b.

The last sentence is only true when $m = n$. Once you have assigned a role to b, b is no longer free to take on other assignments. Use some letter other than m, n, or b in the last phrase. For example, say, "$n = 2c$ for some integer c."

PRECISION OF PHRASING

Standard English allows for a degree of vagueness that is generally unacceptable in the communication of mathematical ideas. Mathematics is a very precise discipline, and it is best to replace vague phrasing with the appropriate specific terminology.

• Avoid the use of imprecise terms.

Example. There is an integer p for which $p^2 + 1$ is prime since 2 makes it 5.

The sentence is unclear; the verb *makes* and the object *it* are both vague. The author should be more specific. For example, it would be preferable to say:

> There is an integer p for which $p^2 + 1$ is prime. Indeed, when $p = 2$, $p^2 + 1 = 5$, which is prime.

Example. For any real number there exists a real number whose sum equals zero.

In order to discuss a sum, we must refer to at least two numbers. The author should rephrase the statement. For example:

For any real number x there exists a real number y such that $x + y = 0$.

Example. Suppose y is a real number between 3 and 5.

Is the author here *including* or *excluding* the possibilities that $y = 3$ and/or $y = 5$? The word "between" is somewhat nebulous. For clarity, it is best to modify "between" as appropriate:

y is *strictly* between 3 and 5

or

y is between 3 and 5 *inclusive.*

Alternatively, the author could avoid the use of "between" entirely by writing

$y \in (3, 5)$ (or $y \in [3, 5]$ as appropriate)

or by writing

$3 < y < 5$ (or $3 \leq y \leq 5$ as appropriate).

Example. Let x and y be real numbers. Then either $x > y$ or $y > x$.

The author has not excluded the possibility that $x = y$, in which case the conclusion does not hold. If the author intends to exclude the case $x = y$, he or she should say:

Let x and y be two *distinct* real numbers.

Otherwise, the author should modify the second sentence to read, for example:

Then either $x > y$ or $y \geq x$.

- Make certain that your use of a term agrees with the definition of that term.

Example. If a is a nonnegative integer, then -a is a negative integer.

The definition of nonnegative is *greater than or equal to zero.* In particular, 0 is a nonnegative integer and $(-0) = 0$ (which is certainly not a negative integer). So the implication, as stated, is false . The author should replace "nonnegative" with "positive."

Example. Since a divides b, $\frac{b}{a}$ = c for some integer c.

By definition, if a divides b, then b = ac for some integer c. However, it does not necessarily follow that $\frac{b}{a} = c$. For example, suppose a = 0 and b = 0. Then a divides b since $0 = 0 \cdot 1$. (Here c = 1.) However, $\frac{b}{a}$ is $\frac{0}{0}$, which is not defined.

Example. Let A = {1, 2, 3}. Then $3 \subseteq A$.

3 is an element of the set A, not a subset of A. The appropriate statement is either

$$3 \in A$$

or

$$\{3\} \subseteq A.$$

Example. $\{x \in \mathbf{R}: x^2 = 4\} = \{x \in \mathbf{R}: x = \sqrt{4}\}$.

These two sets are not equal. $\{x \in \mathbf{R}: x^2 = 4\}$ is the set {-2, 2} while $\{x \in \mathbf{R}: x = \sqrt{4}\}$ consists of the single element 2. The symbol " $\sqrt{\ }$ " is defined to be the positive square root.

- Avoid using the same word to refer to different items.

  Example. "There exists a unique real number y such that $y < 0$ and $y + 3 > 0$" is a false statement since both y = -1 and y = -2 make the statement true. The statement was that only one y value could make the statement true. Since more than one y value makes the statement true, the statement is false.

  The author here has used the word "statement" to refer to different items. The result is confusing. One improvement might be:

  > Consider the statement "There exists a unique real number y such that $y < 0$ and $y + 3 > 0$." Since $-1 < 0$ and $(-1) + 3 > 0$, the value y = -1 satisfies both inequalities. Furthermore, since $-2 < 0$ and $(-2) + 3 > 0$, the value y = -2 also satisfies both inequalities. Hence there are at least two values for y which satisfy both inequalities. Therefore the original statement is false since uniqueness is violated.

STYLISTIC POINTS

Sometimes a proof that is correct and detailed at each step is nonetheless difficult to follow, simply for stylistic reasons. It is worthwhile to make it pleasant for your audience to read your work. You will develop your own style, but there are some general points to keep in mind when writing.

• Make your work legible. In particular, help the reader by spacing your work appropriately on the page and by balancing symbols with words.

Example. Let $x \in \mathbf{Z}$. x even $\Rightarrow x = 2q$ for some $q \in \mathbf{Z}$. Then $x^2 + x = (2q)^2 + (2q) = 4q^2 + 2q = 2(2q^2 + q)$. $q \in \mathbf{Z} \Rightarrow 2q^2 + q \in \mathbf{Z} \Rightarrow x^2 + x$ is even.
x odd $\Rightarrow x = 2k + 1$ for some $k \in \mathbf{Z}$. Then $x^2 + x = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$. $k \in \mathbf{Z} \Rightarrow 2k^2 + 3k + 1 \in \mathbf{Z}$ $\Rightarrow x^2 + x$ is even. Hence $x \in \mathbf{Z} \Rightarrow x^2 + x$ is even.

While this is a correct argument, it is frustrating to read. It is preferable to allow some space and to use words in place of some symbols. For example:

Let x be any integer. If x is even, then $x = 2q$ for some integer q. So

$$\begin{aligned} x^2 + x &= (2q)^2 + 2q \\ &= 4q^2 + 2q \\ &= 2(2q^2 + q). \end{aligned}$$

Since q is an integer, $2q^2 + q$ is an integer as well. Thus $x^2 + x$ is twice an integer and so $x^2 + x$ is even.

If x is odd, then $x = 2k + 1$ for some integer k. Then

$$\begin{aligned} x^2 + x &= (2k + 1)^2 + (2k + 1) \\ &= 4k^2 + 4k + 1 + 2k + 1 \\ &= 2(2k^2 + 3k + 1). \end{aligned}$$

Since k is an integer, $2k^2 + 3k + 1$ is an integer as well and so $x^2 + x$ is twice an integer. Hence $x^2 + x$ is even.

Therefore for any integer x, $x^2 + x$ is even.

• Include transitional phrases to help guide the reader.

Example. Let $x \in \mathbf{R}$ and $y \in \mathbf{R}$. $x^2 \geq 0$ and $y^2 \geq 0$. $x^2 + y^2 \geq 0$. $x^2 + y^2 = -1$ has no real solution.

While each statement follows from the previous, it would be far easier for the reader to follow the argument if the author were to give some guidance. For example:

> Let $x \in \mathbf{R}$ and let $y \in \mathbf{R}$. As x is a real number, $x^2 \geq 0$. As y is a real number, $y^2 \geq 0$. Since x^2 and y^2 are each nonnegative, their sum is nonnegative as well, that is, $x^2 + y^2 \geq 0$. Hence there are no real numbers x and y for which $x^2 + y^2 = -1$.

• If you choose to do a proof by contradiction, contrapositive, or induction, be kind to the reader and indicate your intentions at the beginning of the proof.

CONCLUDING COMMENTS

Good writing is not easy and takes practice. It is rare for someone to write well the first time. Even professional writers constantly revise and rewrite their work. So, *reread* and *rewrite* your paper. It is rare for a first (or even second) draft to achieve the goals laid out above.

The principles discussed in this handout will probably require more thought and care than you have put into your assignments in previous mathematics courses. However, the additional time that you put into your work will make you a clearer thinker and a better student.

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