Homework #8

Solve PDE $\nabla^2 u = 0$

BC $u(0,y) = g_1(y)$

$u(x,0) = u(x,b) = u(a,y) = 0$

Assume $u(x,y) = X(x)\,Y(y)$

Then $\frac{\partial u}{\partial x} = X'Y$, $\frac{\partial^2 u}{\partial x^2} = X''Y$, $\frac{\partial u}{\partial y} = XY'$, $\frac{\partial^2 u}{\partial y^2} = XY''$

and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ becomes

$$X''Y + XY'' = 0, \quad \text{or} \quad \frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Then $\frac{X''}{X} = \frac{-Y''}{Y} = k$, a constant, since LHS, RHS are functions of separate independent variables.

Then $\frac{X''}{X} = k$, or $\boxed{X'' - kX = 0}$, and

$-\frac{Y''}{Y} = k$, or $\boxed{Y'' + kY = 0}$.

Now convert the BCs

$$\left.\begin{aligned} u(x,0) &= X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0 \\ u(x,b) &= X(x)\,Y(b) = 0 \Rightarrow Y(b) = 0 \end{aligned}\right]$$

$u(a,y) = X(a)\,Y(y) = 0 \Rightarrow X(a) = 0$

$u(0,y) = X(0)\,Y(y) = g_1(y)$.

We have two zero boundary conditions on Y, so solve the DE in y first:

DE $Y'' + KY = 0$

BC $Y(0) = Y(b) = 0$

$K = 0$: $Y'' = 0$, so $Y = Ay + B$. $Y(0) = B = 0$, and $Y = Ay$. Now $Y(b) = Ab = 0 \Rightarrow A = 0$ since $b > 0$.
$u = XY = X \cdot 0 = 0$, the trivial solution.

$K > 0$: $\lambda^2 + k = 0$, $\lambda^2 = -k$, $\lambda = \pm i\sqrt{k}$, or
$Y = c_1 \cos\sqrt{k}\,y + c_2 \sin\sqrt{k}\,y$.
$Y(0) = c_1 = 0$, so $Y = c_2 \sin\sqrt{k}\,y$.
$Y(b) = c_2 \sin\sqrt{k}\,b = 0$. If $c_2 = 0$, we get $Y = 0$ and $u = 0$.
Then $\sin\sqrt{k}\,b = 0$, or $\sqrt{k}\,b = n\pi$, $n \in \mathbb{N}$
$\sqrt{k} = \frac{n\pi}{b}$, or $k = \left(\frac{n\pi}{b}\right)^2$, $n \in \mathbb{N}$.

$Y_n = \sin\frac{n\pi y}{b}$.

$K < 0$: $\lambda^2 + k = 0$, so $\lambda = \pm\sqrt{-k}$, $-k > 0$.
Then $Y(y) = c_1 e^{\sqrt{-k}\,y} + c_2 e^{-\sqrt{-k}\,y}$.
$Y(0) = c_1 + c_2 = 0$, so $c_2 = -c_1$
$Y(y) = c_1 \left(e^{\sqrt{-k}\,y} - e^{-\sqrt{-k}\,y}\right)$
$Y(b) = c_1 \left(e^{\sqrt{-k}\,b} - e^{-\sqrt{-k}\,b}\right) = 0 \Rightarrow c_1 = 0$
since the exponential factor $\neq 0$.
Then $Y = 0$ and $u = 0$, the trivial solution.

Solution: $K_n = \left(\frac{n\pi}{b}\right)^2$, $Y_n = \sin\frac{n\pi y}{b}$.

Now we solve $X'' - K_n X = 0$, or $X'' - \left(\frac{n\pi}{b}\right)^2 X = 0$
$\lambda^2 = \left(\frac{n\pi}{b}\right)^2$, or $\lambda = \pm\frac{n\pi}{b}$
Then $X(x) = c_1 e^{\frac{n\pi x}{b}} + c_2 e^{-\frac{n\pi x}{b}}$
$X(a) = 0 = c_1 e^{n\pi\frac{a}{b}} + c_2 e^{-\frac{n\pi a}{b}}$.

Together, we have
$u_n(x,y) = \left(c_{1,n} e^{\frac{n\pi x}{b}} + c_{2,n} e^{-\frac{n\pi x}{b}}\right) \sin\frac{n\pi y}{b}$

Then $u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \left(B_n e^{\frac{n\pi x}{b}} + B_n^* e^{-\frac{n\pi x}{b}} \right)$

Now $u(0,y) = g_1(y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \left(B_n \cdot e^0 + B_n^* e^0 \right)$

$$= \sum_{n=1}^{\infty} (B_n + B_n^*) \sin\frac{n\pi y}{B}.$$

(1) Then $\boxed{B_n + B_n^* = n^{th} \text{ coefficient of the half-range sine series expansion of } g_1(y)}$

and $u(a,y) = 0 = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \left(B_n e^{\frac{n\pi a}{b}} + B_n^* e^{-n\pi\frac{a}{b}} \right)$

Now $\left(B_n e^{\frac{n\pi a}{B}} + B_n^* e^{-\frac{n\pi a}{b}} \right)$ is the n^{th} coefficient of the half-range sine series expansion of $0 = 0$

Hence $\boxed{B_n e^{n\pi\frac{a}{b}} + B_n^* e^{-\frac{n\pi a}{b}} = 0}$, or

(2) $\boxed{B_n^* = -\frac{B_n e^{n\pi\frac{a}{b}}}{e^{-n\pi\frac{a}{b}}} = -B_n e^{2n\pi\frac{a}{b}}}$.

Then we can solve for B_n, B_n^* given (1) and (2).